Trihyperkähler reduction and instanton bundles on $\mathbb{CP}^3$

Marcos Jardim
IMECC - UNICAMP
Departamento de Matemática
Rua Sérgio Buarque de Holanda, 651
13083-859 Campinas, SP, Brazil

Misha Verbitsky
Laboratory of Algebraic Geometry,
Faculty of Mathematics, NRU HSE,
7 Vavilova Str. Moscow, Russia, *and*
Institute for the Physics and Mathematics of the Universe,
University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan

January 15, 2013

Abstract

A trisymplectic structure on a complex $2n$-manifold is a triple of holomorphic symplectic forms such that any linear combination of these forms has rank $2n$, n or 0. We show that a trisymplectic manifold is equipped with a holomorphic 3-web and the Chern connection of this 3-web is holomorphic, torsion-free, and preserves the three symplectic forms. We construct a trisymplectic structure on the moduli of regular rational curves in the twistor space of a hyperkähler manifold, and define a trisymplectic reduction of a trisymplectic manifold, which is a complexified form of a hyperkähler reduction. We prove that the trisymplectic reduction in the space of regular rational curves on the twistor space of a hyperkähler manifold M is compatible with the hyperkähler reduction on M.

As an application of these geometric ideas, we consider the ADHM construction of instantons and show that the moduli space of rank r, charge c framed instanton bundles on $\mathbb{CP}^3$ is a smooth, connected, trisymplectic manifold of complex dimension $4rc$. In particular, it follows that the moduli space of rank 2, charge c instanton bundles on $\mathbb{CP}^3$ is a smooth complex manifold dimension $8c-3$, thus settling part of a 30-year old conjecture.

Contents

1 Introduction **3**
1.1 An overview . . . 3
1.2 3-webs, $SL(2)$-webs and trisymplectic structures . . . 4
1.3 Trisymplectic reduction . . . 6
1.4 Trihyperkähler reduction . . . 6
1.5 Framed instanton bundles on $\mathbb{CP}^3$ . . . 8

2 $SL(2)$-webs on complex manifolds **9**
2.1 $SL(2)$-webs and twistor sections . . . 9
2.2 Hyperkähler manifolds . . . 10
2.3 An example: rational curves on a twistor space . . . 11

3 Trisymplectic structures on vector spaces **12**
3.1 Trisymplectic structures and Mat(2)-action . . . 12
3.2 Trisymplectic structures and invariant quadratic forms on vector spaces with Mat(2)-action . . . 17

4 $SL(2)$-webs and trisymplectic structures **20**
4.1 Trisymplectic structures on manifolds . . . 20
4.2 Chern connection on $SL(2)$-webs and trisymplectic structures . . . 22
4.3 Trisymplectic reduction . . . 23

5 Trihyperkähler reduction **25**
5.1 Hyperkähler reduction . . . 25
5.2 Trisymplectic reduction on the space of twistor sections . . . 26
5.3 Trihyperkähler reduction on the space of twistor sections . . . 27

6 Moment map on twistor sections **29**

7 Trisymplectic reduction and hyperkähler reduction **31**
7.1 The tautological map $\tau: \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}(M/\!\!/\!\!/G)$ . . . 31
7.2 Trihyperkähler reduction and homogeneous bundles on $\mathbb{CP}^1$ . . . 33

8 Case study: moduli spaces of instantons **35**
8.1 Moduli space of framed instantons on $\mathbb{R}^4$ . . . 35
8.2 Moduli space of framed instanton bundles on $\mathbb{CP}^3$ . . . 37
8.3 Moduli space of rank **2** instanton bundles on $\mathbb{CP}^3$ . . . 38

Acknowledgments. The first named author is partially supported by the CNPq grant number 305464/2007-8 and the FAPESP grant number 2005/04558-0. He thanks Amar Henni and Renato Vidal Martins for several discussions related to instanton bundles. The second named author was partially supported by the FAPESP grant number 2009/12576-9, RFBR grant 10-01-93113-NCNIL-a, RFBR grant 09-01-00242-a, AG Laboratory SU-HSE, RF government grant,

ag. 11.G34.31.0023, and Science Foundation of the SU-HSE award No. 10-09-0015. We are grateful to Alexander Tikhomirov for his insight and comments.

1 Introduction

1.1 An overview

In our previous paper [JV], we introduced the notion of holomorphic $SL(2)$-webs, and argued that such manifolds may be regarded as the complexification of hypercomplex manifolds. We showed that manifolds M carrying such structures have a canonical holomorphic connection, called the *Chern connection*, which is torsion-free and has holonomy in $GL(n, \mathbb{C})$, where $\dim_{\mathbb{C}} M = 2n$.

The main example of holomorphic $SL(2)$-webs are given by twistor theory: given a hyperkähler manifold M, then the space of regular holomorphic sections of the twistor fibration $\pi : \mathrm{Tw}(M) \to \mathbb{CP}^1$ is equipped with a holomorphic $SL(2)$-web. We then exploited this fact and the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction of instantons to show that the moduli space of framed instanton bundles on $\mathbb{CP}^3$ is a holomorphic $SL(2)$-web.

The present paper is a sequel to [JV]. Here, we expand in both aspects of our previous paper. On one hand, we describe a new geometric structure on complex manifolds, called *a trisymplectic structure*. The trisymplectic structure is an important special case of a holomorphic $SL(2)$-web. For a trisymplectic structure, we define reduction procedure, allowing us to define a trisymplectic quotient. Applying these new ideas to the ADHM construction of instantons allows us to give a better description of the moduli space of framed instanton bundles on $\mathbb{CP}^3$, and to prove its smoothness and connectness. This allows us to solve a part of a 30-year old conjecture regarding the moduli space of rank 2 instanton bundles on $\mathbb{CP}^3$.

To be more precise, we begin by introducing the notion of *trisymplectic structures on complex manifolds* (see Definition 4.1 below), and show that trisymplectic manifolds carry an induced holomorphic $SL(2)$-web. Our first main goal is to introduce the notion of a *trisymplectic quotient* of a trisymplectic manifold, which would enable us to construct new examples of trisymplectic manifolds out of known ones, e.g. flat ones.

Next, we introduce the notion of *trihyperkähler quotient* $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ for a hyperkähler manifold M, equipped with an action of a Lie group G by considering the trisymplectic quotient of the space $\mathrm{Sec}_0(M)$ of regular holomorphic sections of the twistor fibration of M.

Our first main result (Theorem 5.11) is compatibility between this procedure and the hyperkähler quotient, which we denote by $M/\!\!/\!\!/G$. We show that, under some reasonable conditions, the trihyperkähler reduction $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ admits an open embedding to the space $\mathrm{Sec}_0(M/\!\!/\!\!/G)$ of regular sections of the twistor fibration of the hyperkähler quotient $M/\!\!/\!\!/G$. This shows, in particular, that (similarly to the smoothness of the hyperkähler reduction) the trihyperkähler reduction of M is a smooth trisymplectic manifold.

Our second main result provides an affirmative answer to a long standing conjecture regarding the smoothness and dimension of the moduli space of rank 2 instanton bundles on $\mathbb{CP}^3$, a.k.a mathematical instanton bundles (see Section 8 for precise definitions). More precisely, the moduli space of mathematical instanton bundles with second Chern class (or *charge*) c is conjectured to be an irreducible, nonsingular quasi-projective variety of dimension $8c-3$ (c.f. [CTT, Conjecture 1.2]). The truth of the conjecture for $c \leq 5$ was established by various authors in the past four decades: Barth settled the $c = 1$ case in 1977 [B1]; Hartshorne established the case $c = 2$ in 1978 [H]; Ellingsrud and Stromme settled the $c = 3$ case in 1981 [ES]; the irreducibility of the $c = 4$ case was proved by Barth in 1981 [B2], while the smoothness is due to Le Potier [LeP] (1983); and Coanda–Tikhomirov–Trautmann (2003). More recently, Tikhomirov has shown in [T] that irreducibility holds for odd values of c.

In the present paper, we apply the geometric techniques established above to the ADHM construction of instantons, and show that the moduli space of rank r, charge c *framed* instanton bundles on $\mathbb{CP}^3$ is a smooth, trisymplectic manifold of complex dimension $4rc$ (see Theorem 8.4 below). It then follows easily (see Section 8.3 for the details) that the moduli space of mathematical instanton bundles of charge c is a smooth complex manifold of dimension $8c-3$, thus settling the smoothness part of the conjecture for all values of c.

1.2 3-webs, $SL(2)$-webs and trisymplectic structures

Let M be a real analytic manifold equipped with an atlas $\{U_i \hookrightarrow \mathbb{R}^n\}$ and real analytic transition functions ψ_{ij}. A *complexification* of M is a germ of a complex manifold, covered by open sets $\{V_i \hookrightarrow \mathbb{C}^n\}$ indexed by the same set as $\{U_i\}$, and with the transition function $\psi_{ij}^{\mathbb{C}}$ obtained by analytic extension of ψ_{ij} into the complex domain.

Complexification can be applied to a complex manifold, by considering it as a real analytic manifold first. As shown by Kaledin and Feix (see [F1], [K] and the argument in [JV, Section 1]), a complexification of a real analytic Kähler manifold naturally gives a germ of a hyperkähler manifold. In the paper [JV] we took the next step by looking at a complexification of a hyperkähler manifold. We have shown that such a complexification is equipped with an interesting geometric structure which we called *a holomorphic $SL(2)$-web*.

A holomorphic $SL(2)$-web on a complex manifold M is a collection of involutive holomorphic sub-bundles $S_t \subset TM$, $\operatorname{rk} S_t = \frac{1}{2}\dim M$, parametrized by $t \in \mathbb{CP}^1$, and satisfying the following two conditions. First, $S_t \cap S_{t'} = 0$ for $t \neq t'$, and second, the projector operators $\Pi_{t,t'}$ of TM onto $S_{t'}$ along S_t generate an algebra isomorphic to the matrix algebra $\mathrm{Mat}(2)$ (Definition 2.1).

This structure is a special case of a notion of 3-web developed in 1930-ies by Blaschke and Chern. Let M be an even-dimensional manifold, and S_1, S_2, S_3 a triple of pairwise non-intersecting involutive sub-bundles of TM of dimension $\frac{1}{2}\dim M$. Then S_1, S_2, S_3 is called **a 3-web.** Any 3-web on M gives a rise to a natural connection on TM, called **a Chern connection**. A Chern connection is one which preserves S_i, and its torsion vanishes on $S_1 \otimes S_2$; such a connection

exists, and is unique.

Let $a, b, c \in \mathbb{CP}^1$ be three distinct points. For any $SL(2)$-web, S_a, S_b, S_c is clearly a 3-web. In [JV] we proved that the corresponding Chern connection is torsion-free and holomorphic; also, it is independent from the choice of $a, b, c \in \mathbb{CP}^1$. We also characterized such connections in terms of holonomy, and characterized an $SL(2)$-web in terms of a connection with prescribed holonomy.

Furthermore, we constructed an $SL(2)$-web structure on a component of the moduli space of rational curves on a twistor space of a hyperkähler manifold. By interpreting the moduli space of framed instanton bundles on $\mathbb{CP}^3$ in terms of rational curves on the twistor space of the moduli space of framed bundles on $\mathbb{CP}^2$, we obtained a $SL(2)$-web on the smooth part of the moduli space of framed instanton bundles on $\mathbb{CP}^3$.

In the present paper we explore this notion further, studying those $SL(2)$-webs which appear as moduli spaces of rational lines in the twistor space of a hyperkähler manifold.

It turns out that (in addition to the $SL(2)$-web structure), this space is equipped with the so-called *trisymplectic structure* (see also Definition 4.1).

Definition 1.1. *A* **trisymplectic structure** *on a complex manifold M is a 3-dimensional subspace of $\Omega^2 M$ generated by a triple of holomorphic symplectic forms $\Omega_1, \Omega_2, \Omega_3$, such that any linear combination of $\Omega_1, \Omega_2, \Omega_3$ has rank $n = \dim M$, $\frac{1}{2}n$, or 0.*

In differential geometry, similar structures known as *hypersymplectic structures* were studied by Arnol′d, Atiyah, Hitchin and others (see e.g. [Ar]). The hypersymplectic manifolds are similar to hyperkähler, but instead of quaternions one deals with an algebra $\mathrm{Mat}(2, \mathbb{R})$ of split quaternions. As one passes to complex manifolds and complex-valued holomorphic symplectic forms, the distinction between quaternions and split quaternions becomes irrelevant. Therefore, trisymplectic structures serve as complexifications of both hypersymplectic and hyperkähler structures.

Consider a trisymplectic manifold $(M, \Omega_1, \Omega_2, \Omega_3)$. In Theorem 4.4 we show that the set of degenerate linear combinations of Ω_i is parametrized by $\mathbb{CP}^1$ (up to a constant), and the null-spaces of these 2-forms form an $SL(2)$-web. We also prove that the Chern connection associated with this $SL(2)$-web preserves the 2-forms Ω_i (Theorem 4.6). This allows one to characterize trisymplectic manifolds in terms of the holonomy, similarly as it is done in [JV] with $SL(2)$-webs.

Claim 1.2. *Let M be a complex manifold. Then there is a bijective correspondence between trisymplectic structures on M, and holomorphic connections with holonomy which lies in $G = Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes_{\mathbb{C}} \mathbb{C}^2$ trivially on the second tensor multiplier and in the usual way on $\mathbb{C}^{2n}$.*

Proof: Follows immediately from Theorem 4.6. ∎

1.3 Trisymplectic reduction

In complex geometry, the symplectic reduction is understood as a way of constructing the GIT quotient geometrically. Consider a Kähler manifold M equipped with an action of a compact Lie group G. Assume that G acts by holomorphic isometries, and admits an equivariant moment map $M \xrightarrow{\mu} \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. The **symplectic reduction** $M/\!\!/G$ is the quotient of $\mu^{-1}(0)$ by G. This quotient is a complex variety, Kähler outside of its singular points. When M is projective, one can identify $M/\!\!/G$ with the GIT quotient of M by the action of the complex Lie group.

A hyperkähler quotient is defined in a similar way. Recall that a hyperkähler manifold is a Riemannian manifold equipped with a triple of complex structures I, J, K which are Kähler and satisfy the quaternionic relations. Suppose that a compact Lie group acts on (M, g) by isometries which are holomorphic with respect to I, J, K; such maps are called *hyperkähler isometries*. Suppose, moreover, that there exists a triple of moment maps $\mu_I, \mu_J, \mu_K : M \longrightarrow \mathfrak{g}^*$ associated with the symplectic forms $\omega_I, \omega_J, \omega_K$ constructed from g and I, J, K. The *hyperkähler quotient* ([HKLR]) $M/\!\!/\!\!/G$ is defined as $\big(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)\big)/G$. Similarly to the Kähler case, this quotient is known to be hyperkähler outside of the singular locus.

This result is easy to explain if one looks at the 2-form $\Omega := \omega_J + \sqrt{-1}\omega_K$. This form is holomorphically symplectic on (M, I). Then the complex moment map $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\mu_K$ is holomorphic on (M, I), and the quotient $M/\!\!/\!\!/G := \mu_{\mathbb{C}}^{-1}/\!\!/G_{\mathbb{C}}$ is a Kähler manifold. Starting from J and K instead of I, we construct other complex structures on $M/\!\!/\!\!/G$; an easy linear-algebraic argument is applied to show that these three complex structures satisfy the quaternionic relations.

Carrying this argument a step farther, we repeat it for trisymplectic manifolds, as follows. Let $(M, \Omega_1, \Omega_2, \Omega_3)$ be a trisymplectic manifold, that is, a complex manifold equipped with a triple of holomorphic symplectic forms satisfying the rank conditions of Definition 1.1, and $G_{\mathbb{C}}$ a complex Lie group acting on M by biholomorphisms preserving $\Omega_1, \Omega_2, \Omega_3$. Denote by μ_1, μ_2, μ_3 the corresponding complex moment maps, which are assumed to be equivariant. The *trisymplectic reduction* is the quotient of $\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$ by $G_{\mathbb{C}}$.

Under some reasonable non-degeneracy assumptions, we can show that a trisymplectic quotient is also a trisymplectic manifold (Theorem 4.9).

Notice that since $G_{\mathbb{C}}$ is non-compact, this quotient is not always well-defined. To rectify this, a trisymplectic version of GIT quotient is proposed (Subsection 5.3), under the name of *trihyperkähler reduction*.

1.4 Trihyperkähler reduction

Let M be a hyperkähler manifold, and $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{CP}^1$ its twistor space (Subsection 2.2). A holomorphic section of π is called *regular* if the normal bundle to its image is isomorphic to a sum of $\dim M$ copies of $\mathcal{O}(1)$. Denote by $\operatorname{Sec}_0(M)$ the space of regular sections of π (Definition 2.11).

One may think of $\mathrm{Sec}_0(M)$ as of a complexification of a hyperkähler manifold M. It is the main example of a trisymplectic manifold used in this paper.

The trisymplectic structure on $\mathrm{Sec}_0(M)$ is easy to obtain explicitly. Let L be a complex structure on M induced by the quaternions (Subsection 2.2), and Ω_L the corresponding holomorphic symplectic form on (M,L). Let

$$\mathsf{ev}_L : \ \mathrm{Sec}_0(M) \longrightarrow (M,L)$$

be the *evaluation map* setting a twistor section $s: \ \mathbb{CP}^1 \longrightarrow \mathrm{Tw}(M)$ to $s(L)$ (we use the standard identification of the space of induced complex structures with $\mathbb{CP}^1$). Let $\mathbf{\Omega}$ be a 3-dimensional space of holomorphic 2-forms on $\mathrm{Sec}_0(M)$ generated by $\mathsf{ev}_I^*(\Omega_I)$, $\mathsf{ev}_J^*(\Omega_J)$ and $\mathsf{ev}_K^*(\Omega_K)$. Then $\mathbf{\Omega}$ defines a trisymplectic structure (Claim 5.4).

In this particular situation, the trisymplectic quotient can be defined using a GIT-like construction as follows.

Let G be a compact Lie group acting on a hyperkähler manifold M by hyperkähler isometries. Then G acts on $\mathrm{Sec}_0(M)$ preserving the trisymplectic structure described above. Moreover, there is a natural Kähler metric on $\mathrm{Sec}_0(M)$ constructed in [KV] as follows. The twistor space $\mathrm{Tw}(M)$ is naturally isomorphic, as a smooth manifold, to $M\times\mathbb{CP}^1$. Consider the product metric on $\mathrm{Tw}(M)$, and let $\nu: \ \mathrm{Sec}_0(M) \longrightarrow \mathbb{R}^+$ be a map associating to a complex curve its total Riemannian volume. In [KV] it was shown that ν is a Kähler potential, that is, $dd^c\nu$ is a Kähler form on $\mathrm{Sec}_0(M)$.

Let $\mathbf{\Omega}$ be the standard 3-dimensional space of holomorphic 2-forms on $\mathrm{Sec}_0(M)$,

$$\mathbf{\Omega} = \langle \mathsf{ev}_I^*(\Omega_I), \mathsf{ev}_J^*(\Omega_J), \mathsf{ev}_K^*(\Omega_K)\rangle.$$

Then the corresponding triple of holomorphic moment maps is generated by $\mathsf{ev}_I \circ \mu_I^{\mathbb{C}}$, $\mathsf{ev}_J \circ \mu_J^{\mathbb{C}}$, $\mathsf{ev}_K \circ \mu_K^{\mathbb{C}}$, where $\mu_L^{\mathbb{C}}$ is a holomorphic moment map of (M,L). This gives a description of the zero set of the trisymplectic moment map $\boldsymbol{\mu}$: $\mathrm{Sec}_0(M) \longrightarrow \mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{C}^3$,

As follows from Proposition 5.5, a rational curve $s\in \mathrm{Sec}_0(M)$ lies in $\boldsymbol{\mu}^{-1}(0)$ if and only if s lies in a set of all pairs $(m,t)\in M\times\mathbb{CP}^1 \simeq \mathrm{Tw}(M)$ satisfying $\mu_t^{\mathbb{C}}(m)=0$, where $\mu_t^{\mathbb{C}}: \ (M,t)\longrightarrow \mathfrak{g}^*\otimes_{\mathbb{R}}\mathbb{C}$ is the holomorphic moment map corresponding to the complex structure t.

Now, the zero set $\boldsymbol{\mu}^{-1}(0)$ of the trisymplectic moment map is a Kähler manifold, with the Kähler metric $dd^c\nu$ defined as above. Therefore, one could define the symplectic quotient $\boldsymbol{\mu}^{-1}(0)/\!\!/G$. This quotient, denoted by $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$, is called the *trihyperkähler quotient* of $\mathrm{Sec}_0(M)$ (see Definition 5.9).

One of the main results of the present paper is the following theorem relating the trihyperkähler quotient and the hyperkähler quotient.

Theorem 1.3. *Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M/\!\!/\!\!/G$ is smooth. Then there exists an open embedding $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G \xrightarrow{\Psi} \mathrm{Sec}_0(M/\!\!/\!\!/G)$, which is compatible with the trisymplectic structures on $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ and $\mathrm{Sec}_0(M/\!\!/\!\!/G)$.*

Proof: This is Theorem 5.11. ■

The flatness of M, assumed in Theorem 1.3, does not seem to be necessary, but we were unable to prove it without this assumption.

1.5 Framed instanton bundles on $\mathbb{CP}^3$

In Section 8, the geometric techniques introduced in the previous sections are applied to the study of the moduli space of framed instanton bundles on $\mathbb{CP}^3$.

Recall that a holomorphic vector bundle $E \to \mathbb{CP}^3$ is called an *instanton bundle* if $c_1(E) = 0$ and $H^0(E(-1)) = H^1(E(-2)) = H^2(E(-2)) = H^3(E(-3)) = 0$. The integer $c := c_2(E)$ is called the *charge* of E.

This nomenclature comes from the fact that instanton bundles which are trivial on the lines of the twistor fibration $\mathbb{CP}^3 \to S^4$ (a.k.a. *real lines*) are in 1-1 correspondence, via twistor transform, with non-Hermitian anti-self-dual connections on S^4 (see [JV, Section 3]). Note however that there are instanton bundles which are not trivial on every real line.

Moreover, given a line $\ell \subset \mathbb{P}^3$, a *framing* on E at ℓ is a choice of an isomorphism $\phi : E|_\ell \to \mathcal{O}_\ell^{\oplus \mathrm{rk} E}$. A *framed instanton bundle* is a pair (E, ϕ) consisting of an instanton bundle E restricting trivially to ℓ and a framing ϕ at ℓ. Two framed bundles (E, ϕ) and (E', ϕ') are isomorphic if there exists a bundle isomorphism $\Psi : E \to E'$ such that $\phi' = \phi \circ (\Psi|_\ell)$.

Frenkel and the first named author established in [FJ] a 1-1 correspondence between isomorphism classes of framed instanton bundles on $\mathbb{CP}^3$ and solutions of the *complex ADHM equations* (a.k.a *1-dimensional ADHM equation*) in [J2]).

More precisely, let V and W be complex vector spaces of dimension c and r, respectively, and consider matrices $(k = 1, 2)$ $A_k, B_k \in \mathrm{End}(V)$, $I_k \in \mathrm{Hom}(W, V)$ and $J_k \in \mathrm{Hom}(V, W)$. The 1-dimensional ADHM equations are

$$\begin{cases} [A_1, B_1] + I_1 J_1 = 0 \\ [A_2, B_2] + I_2 J_2 = 0 \\ [A_1, B_2] + [A_2, B_1] + I_1 J_2 + I_2 J_1 = 0 \end{cases} \tag{1.1}$$

One can show [FJ, Main Theorem] that the moduli space of framed instanton bundles on $\mathbb{CP}^3$ coincides with the set of *globally regular* solutions (see Definition 8.1 below) of the 1-dimensional ADHM equations modulo the action of $GL(V)$.

It turns out that the three equations in (1.1) are precisely the three components of a trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}} : \mathrm{Sec}_0(M) \to \mathfrak{u}(V)^* \otimes_{\mathbb{R}} \Gamma(\mathcal{O}_{\mathbb{CP}^1}(2))$ on (an open subset of) a flat hyperkähler manifold M, so that the moduli space of framed instanton bundles coincides with a trihyperkähler reduction of a flat space (Theorem 8.3). It then follows that the moduli space of framed instanton bundles on $\mathbb{CP}^3$ of rank r and charge c is a smooth trisymplectic manifold of dimension $4rc$.

On the other hand, a *mathematical instanton bundle* is a rank 2 stable holomorphic vector bundle $E \to \mathbb{CP}^3$ with $c_1(E) = 0$ and $H^1(E(-2)) = 0$. It is easy to see, using Serre duality and stability, that every mathematical instanton

bundle is a rank 2 instanton bundle. Conversely, every rank 2 instanton bundle is stable, and thus a mathematical instanton bundle. We explore this fact to complete the paper in Section 8.3 by showing how the smoothness of the moduli space of framed rank 2 instanton bundles settles the smoothness part of the conjecture on the moduli space of mathematical instanton bundles.

2 $SL(2)$-webs on complex manifolds

In this section, we repeat basic results about $SL(2)$-webs on complex manifolds. We follow [JV].

2.1 $SL(2)$-webs and twistor sections

The following notion is based on a classical notion of a 3-web, developed in the 1930-ies by Blaschke and Chern, and much studied since then.

Definition 2.1. *Let M be a complex manifold,* $\dim_{\mathbb{C}} M = 2n$, *and $S_t \subset TM$ a family of n-dimensional holomorphic sub-bundles, parametrized by $t \in \mathbb{CP}^1$. This family is called* **a holomorphic $SL(2)$-web** *if the following conditions are satisfied*

(i) *Each S_t is involutive (integrable), that is, $[S_t, S_t] \subset S_t$.*

(ii) *For any distinct points $t, t' \in \mathbb{CP}^1$, the foliations S_t, $S_{t'}$ are transversal: $S_t \cap S_{t'} = \emptyset$.*

(iii) *Let $P_{t,t'} : TM \longrightarrow S_t \hookrightarrow TM$ be a projection of TM to S_t along $S_{t'}$. Then the operators $A(P_{t,t'}) \in \operatorname{End}(TM)$ generate a 3-dimensional sub-bundle in $\operatorname{End}(TM)$, where $A(P_{t,t'}) = P_{t,t'} - \frac{1}{2}\dim_{\mathbb{C}} M\mathbf{1}$ denotes the traceless part of $P_{t,t'}$.*

Since S_t and $S_{t'}$ are mid-dimensional, transversal foliations, it follows that $T_m M = S_t(m) \oplus S_{t'}(m)$ for each point $m \in M$. According to this splitting, $P_{t,t'}(m)$ is simply a projection onto the first factor.

Remark 2.2. *The operators $P_{t,t'} \subset \operatorname{End}(M)$ generate a Lie algebra isomorphic to $\mathfrak{sl}(2)$.*

Definition 2.3. *(see e.g. [A]) Let B be a holomorphic vector bundle over a complex manifold M. A* **holomorphic connection** *on B is a holomorphic differential operator $\nabla : B \longrightarrow B \otimes \Omega^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, for any folomorphic function f on M.*

Remark 2.4. *Let ∇ be a holomorphic connection on a holomorphic bundle, considered as a map $\nabla : B \longrightarrow B \otimes \Lambda^{1,0}M$, and $\bar\partial : B \longrightarrow B \otimes \Lambda^{0,1}M$ the holomorphic structure operator. The sum $\nabla_f := \nabla + \bar\partial$ is clearly a connection. Since ∇ is holomorphic, $\nabla\bar\partial + \bar\partial\nabla = 0$, hence the curvature ∇_f^2 is of type $(2,0)$. The converse is also true: a $(1,0)$-part of a connection with curvature of type $(2,0)$ is always a holomorphic connection.*

Proposition 2.5. *([JV]) Let $S_t, t \in \mathbb{CP}^1$ be an $SL(2)$-web. Then there exists a unique torsion-free holomorphic connection preserving S_t, for all $t \in \mathbb{CP}^1$.* ■

Definition 2.6. *This connection is called* **a Chern connection** *of an $SL(2)$-web.*

Theorem 2.7. *([JV]) Let M be a manifold equipped with a holomorphic $SL(2)$-web. Then its Chern connection is a torsion-free affine holomorphic connection with holonomy in $GL(n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$ as a centralizer of an $SL(2)$-action, where $\mathbb{C}^{2n}$ is a direct sum of n irreducible $GL(2)$-representations of weight 1. Conversely, every connection with such holonomy preserves a holomorphic $SL(2)$-web.* ■

2.2 Hyperkähler manifolds

Definition 2.8. *Let (M, g) be a Riemannian manifold, and I, J, K endomorphisms of the tangent bundle TM satisfying the quaternionic relations*

$$I^2 = J^2 = K^2 = IJK = -\mathbf{1}_{TM}.$$

The triple (I, J, K) together with the metric g is called **a hyperkähler structure** *if I, J and K are integrable and Kähler with respect to g.*

Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ on M:

$$\omega_I(\cdot, \cdot) := g(\cdot, I\cdot), \quad \omega_J(\cdot, \cdot) := g(\cdot, J\cdot), \quad \omega_K(\cdot, \cdot) := g(\cdot, K\cdot). \tag{2.1}$$

An elementary linear-algebraic calculation implies that the 2-form

$$\Omega := \omega_J + \sqrt{-1}\omega_K \tag{2.2}$$

is of Hodge type $(2, 0)$ on (M, I). This form is clearly closed and non-degenerate, hence it is a holomorphic symplectic form.

In algebraic geometry, the word "hyperkähler" is essentially synonymous with "holomorphically symplectic", due to the following theorem, which is implied by Yau's solution of Calabi conjecture ([Bea, Bes]).

Theorem 2.9. *Let M be a compact, Kähler, holomorphically symplectic manifold, ω its Kähler form, $\dim_{\mathbb{C}} M = 2n$. Denote by Ω the holomorphic symplectic form on M. Assume that $\int_M \omega^{2n} = \int_M (\operatorname{Re}\Omega)^{2n}$. Then there exists a unique hyperkähler metric g within the same Kähler class as ω, and a unique hyperkähler structure (I, J, K, g), with $\omega_J = \operatorname{Re}\Omega$, $\omega_K = \operatorname{Im}\Omega$.* ■

Every hyperkähler structure induces a whole 2-dimensional sphere of complex structures on M, as follows. Consider a triple $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, and let $L := aI + bJ + cK$ be the corresponging quaternion. Quaternionic relations imply immediately that $L^2 = -1$, hence L is an almost complex structure. Since I, J, K are Kähler, they are parallel with respect to the Levi-Civita connection. Therefore, L is also parallel. Any parallel complex structure is

integrable, and Kähler. We call such a complex structure $L = aI + bJ + cK$ a *complex structure induced by the hyperkähler structure*. The corresponding complex manifold is denoted by (M, L). There is a 2-dimensional holomorphic family of induced complex structures, and the total space of this family is called the **twistor space** of a hyperkähler manifold; it is constructed as follows.

Let M be a hyperkähler manifold. Consider the product $\operatorname{Tw}(M) = M \times S^2$. Embed the sphere $S^2 \subset \mathbb{H}$ into the quaternion algebra $\mathbb{H}$ as the subset of all quaternions J with $J^2 = -1$. For every point $x = m \times J \in X = M \times S^2$ the tangent space $T_x \operatorname{Tw}(M)$ is canonically decomposed $T_x X = T_m M \oplus T_J S^2$. Identify S^2 with $\mathbb{CP}^1$, and let $I_J : T_J S^2 \to T_J S^2$ be the complex structure operator. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\mathrm{Tw}} \circ I_{\mathrm{Tw}} = -1$. It depends smoothly on the point x, hence it defines an almost complex structure on $\operatorname{Tw}(M)$. This almost complex structure is known to be integrable (see e.g. [Sal]).

Definition 2.10. *The space* $\operatorname{Tw}(M)$ *constructed above is called* **the twistor space** *of a hyperkähler manifold.*

2.3 An example: rational curves on a twistor space

The basic example of holomorphic $SL(2)$-webs comes from hyperkähler geometry. Let M be a hyperkähler manifold, and $\operatorname{Tw}(M)$ its twistor space. Denote by $\operatorname{Sec}(M)$ is the space of holomorphic sections of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{CP}^1$.

We consider $\operatorname{Sec}(M)$ as a complex variety, with the complex structure induced from the Douady space of rational curves on $\operatorname{Tw}(M)$. Clearly, for any $C \in \operatorname{Sec}(M)$, $T_C\operatorname{Sec}(M)$ is a subspace in the space of sections of the normal bundle N_C. This normal bundle is naturally identified with $T_\pi \operatorname{Tw}(M)\big|_C$, where $T_\pi \operatorname{Tw}(M)$ denotes the vertical tangent space.

For each point $m \in M$, one has a horizontal section $C_m := \{m\} \times \mathbb{CP}^1$ of π. The space of horizontal sections of π is denoted $\operatorname{Sec}_{hor}(M)$; it is naturally identified with M. It is easy to check that $NC_m = \mathcal{O}(1)^{\dim M}$, hence some neighbourhood of $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ is a smooth manifold of dimension $2\dim M$. It is easy to see that $\operatorname{Sec}(M)$ is a complexification of $M \simeq \operatorname{Sec}_{hor}(M)$, considered as a real analytic manifold (see [V2]).

Definition 2.11. *A twistor section* $C \in \operatorname{Sec}(M)$ *whose normal bundle* N_C *is isomorphic to* $\mathcal{O}(1)^{\dim M}$ *is called* **regular**.

Let $\operatorname{Sec}_0(M)$ be the subset of $\operatorname{Sec}(M)$ consisting of regular sections. Clearly, $\operatorname{Sec}_0(M)$ is a smooth, Zariski open subvariety in $\operatorname{Sec}(M)$, containing the set $\operatorname{Sec}_{hor}(M)$ of horizontal sections.

The space $\operatorname{Sec}_0(M)$ admits the structure of a holomorphic $SL(2)$-web, constructed as follows. For each $C \in \operatorname{Sec}_0(M)$ and $t \in \mathbb{CP}^1 = C$, define $S_t \subset TC = \Gamma_C(N_C)$ as the space of all sections of N_C vanishing at $t \in C$.

It is not difficult to check that this is a holomorphic $SL(2)$-web. Transversality of S_t and $S_{t'}$ is obvious, because a section of $\mathcal{O}(1)$ vanishing at two points is zero. Integrability of S_t is also clear, since the leaves of S_t are fibers of the evaluation map $ev_t: \operatorname{Sec}(M) \longrightarrow \operatorname{Tw}(M)$, mapping $C: \mathbb{CP}^1 \longrightarrow \operatorname{Tw}(M)$ to $C(t)$. The last condition follows from the fact that $\Gamma_{\mathbb{CP}^1}(V \otimes_{\mathbb{C}} \mathcal{O}(1)) \simeq V \otimes_{\mathbb{C}} \mathbb{C}^2$, and the projection maps $P_{t,t'}$ act on $V \otimes_{\mathbb{C}} \mathbb{C}^2$ only through the second component.

The space $\operatorname{Sec}_0(M)$ is the main example of an $SL(2)$-web manifold we consider in this paper.

3 Trisymplectic structures on vector spaces

3.1 Trisymplectic structures and $\operatorname{Mat}(2)$-action

This section is dedicated to the study of the following linear algebraic objects, which will be the basic ingredient in the new geometric structures we will introduce later.

Definition 3.1. *Let $\mathbf{\Omega}$ be a 3-dimensional space of complex linear 2-forms on a complex vector space V. Assume that*

(i) *$\mathbf{\Omega}$ contains a non-degenerate form;*

(ii) *For each non-zero degenerate $\Omega \in \mathbf{\Omega}$, one has $\operatorname{rk}\Omega = \frac{1}{2}\dim V$.*

Then $\mathbf{\Omega}$ is called **a trisymplectic structure on** V*, and* $(V, \mathbf{\Omega})$ **a trisymplectic space.**

Remark 3.2. *If V is not a complex, but a real vector space, this notion defines either a quaternionic Hermitian structure, or a structure known as* **hypersymplectic** *and associated with a action of split quaternions, c.f. [Ar].*

Lemma 3.3. *Let $(V, \mathbf{\Omega})$ be a be a trisymplectic space, and $\Omega_1, \Omega_2 \in \mathbf{\Omega}$ two non-zero, degenerate forms which are not proportional. Then the annihilator $\operatorname{Ann}(\Omega_1)$ does not intersect $\operatorname{Ann}(\Omega_2)$.*

Proof: Indeed, if these two spaces intersect in a subspace $C \subset V$, strictly contained in $\operatorname{Ann}(\Omega_1)$, some linear combination of Ω_1 and Ω_2 would have annihilator C, which is impossible, because $0 < \dim C < \frac{1}{2}\dim V$. If $\operatorname{Ann}(\Omega_1) = \operatorname{Ann}(\Omega_2)$, we could consider Ω_1, Ω_2 as non-degenerate forms $\Omega_1\big|_W, \Omega_2\big|_W$ on $W := V/\operatorname{Ann}(\Omega_2)$, which are obviously not proportional. We interpret $\Omega_i\big|_W$ as a bijective map from W to W^*. Let w be an eigenvector of an operator $\Omega_1\big|_W \circ \left(\Omega_2\big|_W\right)^{-1} \in \operatorname{End}(W)$, and λ its eigenvalue. Then $\Omega_1(w,x) = \lambda\Omega_2(w,x)$, for each $x \in W$, hence w lies in the annihilator of $\Omega_1\big|_W - \lambda\Omega_2\big|_W$. Then $\Omega_1 - \lambda\Omega_2$ has an annihilator strictly larger than $\operatorname{Ann}(\Omega_2)$, which is impossible, unless $\Omega_1 = \lambda\Omega_2$. ■

Given two non-proportional, degenerate forms $\Omega_1, \Omega_2 \in \mathbf{\Omega}$, one has that $V = \operatorname{Ann}(\Omega_1) \oplus \operatorname{Ann}(\Omega_2)$ by the previous Lemma. Thus one can consider projection operators Π_{Ω_1,Ω_2} of V onto $\operatorname{Ann}(\Omega_1)$ along $\operatorname{Ann}(\Omega_2)$.

Proposition 3.4. *Let $(V, \mathbf{\Omega})$ be a be a trisymplectic vector space, and let $H \subset \operatorname{End}(V)$ be the subspace generated by projections Π_{Ω_1,Ω_2} for all pairs of non-proportional, degenerate forms $\Omega_1, \Omega_2 \in \mathbf{\Omega}$. Then*

(A) *H is a subalgebra of $\operatorname{End}(V)$, isomorphic to the matrix algebra $\operatorname{Mat}(2)$.*

(B) *Let $\mathfrak{g} \subset \operatorname{End}(V)$ be a Lie algebra generated by the commutators $[H, H]$, $\mathfrak{g} \cong \mathfrak{sl}(2)$. Then the space $\mathbf{\Omega} \subset \Lambda^2(V)$ is $\mathfrak{g}$-invariant, under the natural action of the Lie algebra $\mathfrak{g}$ on $\Lambda^2 V$.*

(C) *There exists a non-degenerate, $\mathfrak{g}$-invariant quadratic form Q on $\mathbf{\Omega}$, unique up to a constant, such that $\Omega \in \mathbf{\Omega}$ is degenerate if and only if $Q(\Omega, \Omega) = 0$.*

Proof. We begin by establishing item (A); the proof is divided in three steps.

Step 1: We prove that the space $H \subset \operatorname{End}(V)$ is an algebra, and satisfies $\dim H \leq 4$.

Let $\Omega_1, \Omega_2 \in \mathbf{\Omega}$ be two forms which are not proportional, and assume Ω_2 is non-degenerate. Consider an operator $\phi_{\Omega_1,\Omega_2} \in \operatorname{End}(V)$, $\phi_{\Omega_1,\Omega_2} := \Omega_1 \circ \Omega_2^{-1}$, where Ω_1, Ω_2 are understood as operators from V to V^*. As in the proof of Lemma 3.3, consider an eigenvector v of ϕ_{Ω_1,Ω_2}, with the eigenvalue λ. Then $\Omega_1(v, x) = \lambda\Omega_2(v, x)$, for each $x \in V$, hence v lies in the annihilator of $\Omega := \Omega_1 - \lambda\Omega_2$. Since Ω_i are non-proportional, Ω is non-zero, hence $\operatorname{\mathsf{rk}} \Omega = \frac{1}{2} \dim V$. This implies that each eigenspace of ϕ_{Ω_1,Ω_2} has dimension $\frac{1}{2} \dim V$. Choosing another eigenvalue λ' and repeating this procedure, we obtain a 2-form $\Omega' := \Omega_1 - \lambda'\Omega_2$, also degenerate. Let S, S' be annihilators of Ω, Ω', and $\Pi_{S,S'}, \Pi_{S',S}$ be the projection of V onto S or S' along S' or S, respectively. It follows that

$$\phi_{\Omega_1,\Omega_2} = \lambda\Pi_{S',S} + \lambda'\Pi_{S,S'}. \tag{3.1}$$

and ϕ_{Ω_1,Ω_2} can be expressed in an appropriate basis by the matrix

$$\phi_{\Omega_1,\Omega_2} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda' \end{pmatrix}. \tag{3.2}$$

From (3.1) it is clear that the space H is generated by all ϕ_{Ω_1,Ω_2}. It is also clear that when Ω_2 is also non-degenerate, the operator ϕ_{Ω_2,Ω_1} can be expressed as a linear combination of ϕ_{Ω_1,Ω_2} and $\phi_{\Omega_1,\Omega_1} = \mathbf{1}_V$.

Since non-degenerate forms constitute a dense open subset of $\mathbf{\Omega}$, one can choose a basis $\Omega_1, \Omega_2, \Omega_3$ consisting of non-degenerate forms. Since $\phi_{\Omega_i,\Omega}$ is

expressed as a linear combination of ϕ_{Ω,Ω_i} and $\mathbf{1}_V$, and ϕ_{Ω,Ω_i} is linear in Ω, the vector space H is generated by $\phi_{\Omega_i,\Omega_j}, i<j$, and $\mathbf{1}_V$. Therefore, H is at most 4-dimensional. From (3.1) it is clear that for any non-degenerate Ω_1, Ω_2, the operator ϕ_{Ω_1,Ω_2} can be expressed through $\phi_{\Omega_2,\Omega_1} = \phi^{-1}_{\Omega_1,\Omega_2}$ and $\mathbf{1}$:

$$\phi_{\Omega_2,\Omega_1} = a\phi_{\Omega_2,\Omega_1} + b\mathbf{1}. \tag{3.3}$$

Since

$$\phi_{\Omega_i,\Omega_j} \circ \phi_{\Omega_j,\Omega_k} = \phi_{\Omega_i,\Omega_k}, \tag{3.4}$$

the space H is a subalgebra in $\operatorname{End}(V)$ (to multiply some of ϕ_{Ω_i,Ω_j} and $\phi_{\Omega_{i'},\Omega_{j'}}$, you would have to reverse the order when necessary, using (3.3), and then apply (3.4)).

Step 2: We prove that any element of H has form $\phi_{\Omega,\Omega'} + c\mathbf{1}_V$, for some $\Omega, \Omega' \in \mathbf{\Omega}$. Indeed, as we have shown, a general element of H has form

$$h = a\phi_{\Omega_1,\Omega_2} + b\phi_{\Omega_1,\Omega_3} + c\phi_{\Omega_2,\Omega_3} + d\mathbf{1}_V, \tag{3.5}$$

where $\Omega_1, \Omega_2, \Omega_3$ is a basis of non-degenerate forms for $\mathbf{\Omega}$. Since ϕ is linear in the first argument, this gives

$$h = a\phi_{\Omega_1,\Omega_2} + \phi_{b\Omega_1+c\Omega_2,\Omega_3} + dc\mathbf{1}_V. \tag{3.6}$$

If the form $b\Omega_1 + c\Omega_2$ is non-degenerate, we use the reversal as indicated in (3.3), obtaining

$$\phi_{b\Omega_1+c\Omega_2,\Omega_3} = \lambda\phi_{\Omega_3,b\Omega_1+c\Omega_2} + \lambda'\mathbf{1}_V,$$

write, similarly,

$$a\phi_{\Omega_1,\Omega_2} = \mu\phi_{\Omega_1,b\Omega_1+c\Omega_2} + \mu'\lambda'\mathbf{1}_V,$$

then, adding the last two formulae, obtain

$$h = (\mu+1)\phi_{\Omega_1+\Omega_3,b\Omega_1+c\Omega_2} + (\lambda' + \mu' + d)\mathbf{1}_V.$$

We denote the term $b\Omega_1 + c\Omega_2$ in (3.6) as $\Omega(h, \Omega_1, \Omega_2, \Omega_3)$, considering it as a function of h and the basis Ω_i. If $b\Omega_1 + c\Omega_2$ is degenerate, we have to make a different choice of the basis $\Omega_1, \Omega_2, \Omega_3$, in such a way that $\Omega(h, \Omega_1, \Omega_2, \Omega_3)$ becomes non-degenerate. This is done as follows.

If we replace Ω_1 by $\Omega_1' = \Omega_1 + \epsilon\Omega_3$, in the expression (3.5) we get

$$h = a\phi_{\Omega_1',\Omega_2} + b\phi_{\Omega_1',\Omega_3} + c\phi_{\Omega_2,\Omega_3} + \epsilon\phi_{\Omega_3,\Omega_2} + (d+\epsilon)\mathbf{1}_V.$$

Let $\phi_{\Omega_3,\Omega_2} = \lambda\phi_{\Omega_2,\Omega_3} + \alpha\mathbf{1}_V$, as in (3.3). Then

$$\Omega(h, \Omega_1', \Omega_2, \Omega_3) = b\Omega_1' + (c - \epsilon\lambda)\Omega_2 = b\Omega_1 + b\epsilon\Omega_3 + (c-\epsilon)\lambda\Omega_2$$

The difference of these two terms is expressed as

$$\Omega(h, \Omega_1', \Omega_2, \Omega_3) - \Omega(h, \Omega_1, \Omega_2, \Omega_3) = b\epsilon\Omega_3 - \epsilon\lambda\Omega_2.$$

If $\Omega(h, \Omega_1', \Omega_2, \Omega_3)$ remains degenerate for all ϵ, then $b\epsilon\Omega_3 - \epsilon\lambda\Omega_2$ is proportional to $b\Omega_1 + c\Omega_2$, which is impossible, because Ω_i are linearly independent. Therefore, Ω_i can be chosen in such a way that $\Omega(h, \Omega_1, \Omega_2, \Omega_3)$ is non-degenerate, and for such a basis, h is expressed as above.

Step 3: We prove that the algebra H is isomorphic to $\mathrm{Mat}(2)$. Consider the form $B(h_1, h_2) := \mathrm{Tr}(h_1 h_2)$ on H. From Step 2 and (3.1) it follows any element of h can be written as

$$h = \lambda\Pi_{S',S} + \lambda'\Pi_{S,S'}. \tag{3.7}$$

where $\Pi_{S,S'}$ are projection operators. Then,

$$\mathrm{Tr}\, h = B(h, \mathbf{1}_V) = \frac{1}{2}\dim M(\lambda + \lambda'),$$

This is non-zero unless $\lambda = -\lambda'$, and in the latter case $B(h,h) = \lambda^2 \dim M \neq 0$ (unless h vanishes). Therefore, the form B is non-degenerate. Since the Lie algebra $(H, [\cdot,\cdot])$ admits a non-degenerate invariant quadratic form, it is reductive. Since $\dim H \leq 4$, and it has a non-trivial center generated by $\mathbf{1}_V$, it follows from the classification of reductive algebras that either $(H, [\cdot,\cdot]) \cong \mathfrak{sl}(2) \oplus \mathbb{C}\cdot\mathbf{1}_V$ or H is commutative. In the first case, H is obviously isomorphic to $\mathrm{Mat}(2)$. Therefore, to prove that $H \cong \mathrm{Mat}(2)$ it suffices to show that H is not commutative. In the latter case, there exists a basis in V for which all elements of H are upper triangular. However, from (3.2) it would follow that in this case $\dim H = 2$, hence $\dim \mathbf{\Omega} \leq 2$, which contradicts our hypothesis. We have therefore proved Proposition 3.4 (A).

We now consider the second part of the Proposition. By definition, the algebra H is generated as a linear space by idempotents, that is, projection operators. Consider an idempotent $\Pi \in H$. To prove that the Lie algebra $\mathfrak{g} = [H, H]$ preserves $\mathbf{\Omega}$, it would suffice to show that for each $\Omega \in \mathbf{\Omega}$, the form

$$\Pi(\Omega) := \Omega(\Pi(\cdot), \cdot) + \Omega(\cdot, \Pi(\cdot))$$

belongs to $\mathbf{\Omega}$. Since $\Pi(\Omega)$ vanishes on $\ker\Pi$ and is equal to Ω on $\mathrm{Im}\ \Pi$, one has also

$$\Pi(\Omega) = 2\Omega(\Pi(\cdot), \Pi(\cdot)). \tag{3.8}$$

Let ϕ_{Ω_1,Ω_2} be an operator satisfying $\Pi = \lambda\phi_{\Omega_1,\Omega_2} + \lambda'\mathbf{1}_V$ (the existence of such an operator was shown in Step 2). Then $\Pi = \lambda\phi_{\frac{\lambda'}{\lambda}\Omega_2 + \Omega_1, \Omega_2}$. Denote by Ω' the 2-form $\frac{\lambda'}{\lambda}\Omega_2 + \Omega_1$. Clearly, $\mathrm{Ann}(\Omega') = \ker\Pi$. To prove that $\Pi(\Omega) \in \mathbf{\Omega}$ it would suffice to show that $\Pi(\Omega)$ is proportional to Ω'.

Since any linear combination of Ω and Ω' has rank 0, $\dim V$ or $\frac{1}{2}\dim V$, one has $\Omega\big|_{\mathrm{Im}\ \Pi} = \alpha\Omega'\big|_{\mathrm{Im}\ \Pi}$, for some $\alpha \in \mathbb{C}$ (see the proof of Lemma 3.3 for the first use of this argument). Therefore, $\Pi(\Omega)$ is equal to $\alpha\Omega'$ on $\mathrm{Im}\ \Pi$. Also, both of these forms vanish on $\ker\Pi$. Therefore, (3.8) implies that $\Pi(\Omega) = \frac{\alpha}{2}\Omega'$. We proved Proposition 3.4 (B).

To prove item (C), consider the function $\det: \operatorname{Mat}(2) \longrightarrow \mathbb{C}$, and let B be the corresponding quadratic polynomial on $H \cong \operatorname{Mat}(2)$. Since the isomorphism $H \cong \operatorname{Mat}(2)$ is unique up to a twist with $SL(2)$, the function B is well defined.

Let $\Omega \in \mathbf{\Omega}$ be a non-degenerate symplectic form, and $Q_\Omega: \mathbf{\Omega} \longrightarrow \mathbb{C}$ a function mapping Ω' to $B(\phi_{\Omega',\Omega})$. Since ϕ is linear in the first argument, Q_Ω is a quadratic polynomial function, and its zero-set Z coincides with the set of degenerate forms in $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is generated by degenerate forms (see Proposition 3.4, Step 1), the set Z can be either a union of two planes or a zero set a non-degenerate quadratic polynomial. However, since V is equipped with an action of $SL(2)$ preserving $\mathbf{\Omega}$ (Proposition 3.4 (B)), Z cannot be a union of two planes. Therefore, there exists a unique (up to a constant) non-degenerate $SL(2)$-invariant quadratic form on $\mathbf{\Omega}$ vanishing on all degenerate forms.

We proved Proposition 3.4. ■

In a similar way, we obtain the following useful corollary.

Claim 3.5. *Let $(V, \mathbf{\Omega})$ be a trisymplectic space, and $W \subset V$ a complex subspace. Then $\mathbf{\Omega}\big|_W$ is a trisymplectic space if and only if the following two assumptions hold.*

(i) *The space W is H-invariant, where $H \cong \operatorname{Mat}(2)$ is the subalgebra of $\operatorname{End}(V)$ constructed in Proposition 3.4.*

(ii) *A general 2-form $\Omega \in \mathbf{\Omega}$ is non-degenerate on W.*

Proof: Let $Z \subset H$ be the set of idempotents in H. Consider the standard action of $\mathfrak{g} \cong \mathfrak{sl}(2)$ on V constructed in Proposition 3.4. Clearly, V is a direct sum of several 2-dimensional irreducible representations of $\mathfrak{sl}(2)$.

It is easy to see that for every $\Pi \in Z$ there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that Π is a projection of V onto one of two weight components of the weight decomposition associated with $\mathfrak{h}$. If $W \subset V$ is an H-submodule, then $\Pi\big|_W$ is a projection to a weight component $W_0 \subset W$ of dimension $\frac{1}{2}\dim W$. From (3.1) it is also clear that for any degenerate form $\Omega \in \mathbf{\Omega}$, an annihilator of a restriction $\Omega\big|_W$ is equal to the weight component W_0, for an appropriate choice of Cartan subalgebra. Therefore,

$$\dim\left(\operatorname{Ann}\Omega\big|_W\right) = \frac{1}{2}\dim W.$$

Similarly, (3.1) implies that a non-degenerate form is restricted to a non-degenerate form. We obtain that the restriction $\mathbf{\Omega}\big|_W$ to an H-submodule is always a trisymplectic structure on W.

To obtain the converse statement, take two non-degenerate, non-collinear forms $\Omega_1, \Omega_2 \in \mathbf{\Omega}$, and notice that there exist precisely two distinct numbers $t = \lambda, \lambda'$ for which $\Omega_1 + t\Omega_2$ is degenerate (see Proposition 3.4, Step 1). Let S, S' be the corresponding annihilator spaces. As follows from Proposition 3.4, Step 2, H is generated, as a linear space, by the projection operators $\Pi_{S,S'}$,

projecting V to S along S'. For any $W \subset V$ such that the restriction $\mathbf{\Omega}\big|_W$ is trisymplectic, one has $W = S \cap W \oplus S' \cap W$, hence $\Pi_{S,S'}$ preserves W. Therefore, W is an H-submodule. ■

Definition 3.6. *Let $(V, \mathbf{\Omega})$ be a trisymplectic space, and $W \subset V$ a vector subspace. Consider the action of $H \simeq \mathrm{Mat}(2)$ on V induced by the trisymplectic structure. A subspace $W \subset V$ is called* **non-degenerate** *if the subspace $H \cdot W \subset V$ is trisymplectic.*

Remark 3.7. *By Claim 3.5, W is non-degenerate if and only if the restriction of Ω to $H \cdot W$ is non-degenerate for some $\Omega \in \mathbf{\Omega}$.*

3.2 Trisymplectic structures and invariant quadratic forms on vector spaces with $\mathrm{Mat}(2)$-action

Let V be a complex vector space with a standard action of the matrix algebra $\mathrm{Mat}(2)$, i.e. $V \cong V_0 \otimes \mathbb{C}^2$ and $\mathrm{Mat}(2)$ acts only through the second factor. An easy way to obtain the trisymplectic structure is to use non-degenerate, invariant quadratic forms on V.

Consider the natural $SL(2)$-action on V induced by $\mathrm{Mat}(2)$, and extend it multiplicatively to all tensor powers of V. Let $g \in \mathrm{Sym}^2_{\mathbb{C}}(V)$ be an $SL(2)$-invariant, non-degenerate quadratic form on V, and let $\{I, J, K\}$ be a quaternionic basis in $\mathrm{Mat}(2)$, i.e. $\{\mathbf{1}_V, I, J, K\}$ is a basis for $\mathrm{Mat}(2)$ and $I^2 = J^2 = K^2 = IJK = -1$. Then

$$g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y)$$

hence the form $\Omega_I(\cdot,\cdot) := g(\cdot, I\cdot)$ is a symplectic form, obviously non-degenerate; similarly, the forms $\Omega_J(\cdot,\cdot) := g(\cdot, J\cdot)$ and $\Omega_K(\cdot,\cdot) := g(\cdot, K\cdot)$ have the same properties. It turns out that this construction gives a trisymplectic structure, and all trisymplectic structures can be obtained in this way.

Theorem 3.8. *Let V be a vector space equipped with a standard action of the matrix algebra $\mathrm{Mat}(2) \xrightarrow{\rho} \mathrm{End}(V)$, and $\{I, J, K\}$ a quaternionic basis in $\mathrm{Mat}(2)$. Consider the corresponding action of $SL(2)$ on the tensor powers of V. Then*

(i) *Given a non-degenerate, $SL(2)$-invariant quadratic form $g \in \mathrm{Sym}^2(V)$, consider the space $\mathbf{\Omega} \subset \Lambda^2 V$ generated by the symplectic forms $\Omega_I, \Omega_J, \Omega_K$ defined as above,*

$$\Omega_I(\cdot,\cdot) := g(\cdot, I\cdot), \quad \Omega_J(\cdot,\cdot) := g(\cdot, J\cdot), \quad \Omega_K(\cdot,\cdot) := g(\cdot, K\cdot). \tag{3.9}$$

Then $\mathbf{\Omega}$ is a trisymplectic structure on V, with the operators $\Omega_K^{-1} \circ \Omega_J$ and $\Omega_K^{-1} \circ \Omega_I$, generating the algebra $H \cong \mathrm{Mat}(2) := \mathrm{Im}\,(\rho) \subset \mathrm{End}(V)$ as in Proposition 3.4.

(ii) *Conversely, for each trisymplectic structure $\mathbf{\Omega}$ inducing the action of $H \cong \mathrm{Mat}(2)$ on V given by ρ, there exists a unique (up to a constant) $SL(2)$-invariant non-degenerate quadratic form g inducing $\mathbf{\Omega}$ as in* (3.9).

Proof: First, consider the 3-dimensional subspace of $\Lambda^2 V$ generated by $\Omega_I, \Omega_J, \Omega_K$. Regard Ω_I as an operator from V to V^*, $x \mapsto \Omega_I(x, \cdot)$, and similarly for Ω_J and Ω_K; let $h := \Omega_K^{-1} \circ \Omega_J \in \mathrm{End}(V)$. Then

$$h(x) = \Omega_K^{-1}(-g(Jx, \cdot)) = -KJx = Ix,$$

hence $h = I$. Similarly, one concludes that $\Omega_K^{-1} \circ \Omega_I = J$, hence $\Omega_K^{-1} \circ \Omega_J$ and $\Omega_K^{-1} \circ \Omega_I$ generate H as an algebra.

To complete the proof of the first claim of the Theorem, it remains for us to show that $\mathbf{\Omega}$ is a trisymplectic structure. For this it would suffice to show that any non-zero, degenerate form $\Omega \in \mathbf{\Omega}$ has rank $\frac{1}{2}\dim V$. Consider V as a tensor product $V = V_0 \otimes \mathbb{C}^2$, with $\mathrm{Mat}(2)$ acting on the second factor. Choose a basis $\{x, y\}$ in $\mathbb{C}^2$, so that $V = V_0 \otimes x \oplus V_0 \otimes y$. From $SL(2)$-invariance it is clear that $g(v_0 \otimes \zeta) = g(v_0 \otimes \xi)$ for any non-zero $\zeta, \xi \in \mathbb{C}^2$. Therefore, $V_0 \otimes x \subset V$ and $V_0 \otimes y \subset V$ are isotropic subspaces, dual to each other. Denote by Ω_{V_0} the corresponding bilinear form on V_0:

$$\Omega_{V_0}(v, v') := g(v \otimes x, v' \otimes y).$$

Since the group $SL(2)$ acts transitively on the set of all $\zeta, \xi \in \mathbb{C}^2$ satisfying $\zeta \wedge \xi = x \wedge y$, we obtain

$$\Omega_{V_0}(v, v') = g(v \otimes x, v' \otimes y) = -g(v \otimes y, v' \otimes x) = -\Omega_{V_0}(v', v).$$

Therefore, Ω_{V_0} is skew-symmetric. Conversely, g can be expressed through Ω_{V_0}, as follows. Given $x', y' \in \mathbb{C}^2$ such that $x' \wedge y' \neq 0$, and $v \otimes x_1, w \otimes y_1 \in V$, we find $h \in SL(2)$ such that $h(x') = \lambda x$ and $h(y') = \lambda y$ with $\lambda = \frac{x_1 \wedge y_1}{x \wedge y}$. Since g is $SL(2)$-invariant, one has

$$g(v \otimes x', w \otimes y') = \lambda^2 g(v \otimes x, w \otimes y).$$

Therefore, for appropriate symplectic form $\Omega_{\mathbb{C}^2}$ on $\mathbb{C}^2$, one would have

$$g(v \otimes x', w \otimes y') = \Omega_{V_0}(v, w) \cdot \Omega_{\mathbb{C}^2}(x', y'). \tag{3.10}$$

This gives us a description of the group $\mathsf{St}(H, g) \subset \mathrm{End}(V)$ which fixes the algebra $H \subset \mathrm{End}(V)$ and g. Indeed, from (3.10), we obtain that $\mathsf{St}(H, g) \cong \mathrm{Sp}(V_0, \Omega_{V_0})$ acting on $V = V_0 \otimes \mathbb{C}^2$ in a standard way, i.e. trivially on the second factor.

Since all elements of $\mathbf{\Omega}$ are by construction fixed by $\mathsf{St}(H, g) \cong \mathrm{Sp}(V_0, \Omega_{V_0})$, for any $\Omega \in \mathbf{\Omega}$, the annihilator of Ω is $\mathrm{Sp}(V_0, \Omega_{V_0})$-invariant. However, $V \cong V_0 \oplus V_0$ is isomorphic to a sum of two copies of the fundamental representation of $\mathrm{Sp}(V_0, \Omega_{V_0})$, hence any $\mathrm{Sp}(V_0, \Omega_{V_0})$-invariant space has dimension $0, \frac{1}{2}\dim V$, or $\dim V$. We finished the proof of Theorem 3.8 (i).

The proof of the second part of the Theorem is divided into several steps.

Step 1. Let $I \in \operatorname{Mat}(2)$ be such that $I^2 = -\mathbf{1}_V$. Consider the action $\rho_I : U(1) \longrightarrow \operatorname{End}(V)$ generated by $t \longrightarrow \cos t \mathbf{1}_V + \sin t \rho(I)$. As shown in Proposition 3.4 (B), $\mathbf{\Omega}$ is an $SL(2)$-subrepresentation of $\Lambda^2 V$. This representation is by construction irreducible. Since it is 3-dimensional, it is isomorphic to the adjoint representation of $SL(2)$; let $\phi : \mathfrak{sl}(2) \longrightarrow \mathbf{\Omega}$ be an isomorphism. Therefore, there exists a 2-form $\Omega_I \in \mathbf{\Omega}$ fixed by the action of ρ_I, necessarily unique up to a constant multiplier. Write $g_I(x,y) := -\Omega_I(x, Iy)$. Then

$$g_I(y,x) = \Omega_I(y, Ix) = -\Omega_I(Ix, y) = -\Omega_I(I^2x, Iy) = \Omega(x, Iy) = g_I(x,y),$$

hence g_I is symmetric, i.e. $g_I \in \operatorname{Sym}^2_{\mathbb{C}}(V)$.

Step 2. Now let $\{I, J, K\}$ be the quaternionic basis for $\operatorname{Mat}(2)$. We prove that the symmetric tensor g_I constructed in Step 1 is fixed by the subgroup $\{\pm 1, \pm I, \pm J, \pm K\} \subset SL(2) \subset \operatorname{Mat}(2)$, for an appropriate choice of $\Omega_I \in \mathbf{\Omega}$.

Using the $SL(2)$-invariant isomorphism $\phi : \mathfrak{sl}(2) \longrightarrow \mathbf{\Omega}$ constructed in Step 1, and the identification of $\mathfrak{sl}(2)$ with the subspace of $\operatorname{Mat}(2)$ generated by I, J and K, we fix a choice of Ω_I by requiring that $\phi(I) = \Omega_I$. Then, J and K, considered as elements of $SL(2)$, act on Ω_I by -1:

$$\Omega_I(J\cdot, J\cdot) = -\Omega_I(\cdot,\cdot), \quad \Omega_I(K\cdot, K\cdot) = -\Omega_I(\cdot,\cdot).$$

This gives

$$g_I(J\cdot, J\cdot) = \Omega_I(J\cdot, IJ\cdot) = -\Omega_I(J\cdot, JI\cdot) = \Omega_I(\cdot, I\cdot) = g(\cdot,\cdot).$$

We have shown that J, considered as an element of $SL(2)$, fixes g_I. The same argument applied to K implies that K also fixes g_I. We have shown that g_I is fixed by the Klein subgroup $\mathfrak{K} := \{\pm 1, \pm I, \pm J, \pm K\} \subset SL(2)$.

Step 3. We prove that g_I is $SL(2)$-invariant.

Consider $\operatorname{Sym}^2 V$ as a representation of $SL(2)$. Since V is a direct sum of weight 1 representations, Clebsch-Gordon theorem implies that $\operatorname{Sym}^2 V$ is a sum of several weight 2 and trivial representations. However, no element on a weight 2 representation can be $\mathfrak{K}$-invariant. Indeed, a weight 2 representation W_2 is isomorphic to an adjoint representation, that is, a complex vector space generated by the imaginary quaternions: $W_2 := \langle I, J, K\rangle \subset \operatorname{Mat}(2)$ Clearly, no non-zero linear combination of I, J, K. can be $\mathfrak{K}$-invariant. Since g_I is $\mathfrak{K}$-invariant, this implies that g_I lies in the $SL(2)$-invariant part of $\operatorname{Sym}^2_{\mathbb{C}} V$.

Step 4. We prove that g_I is proportional to $g_{I'}$, for any choice of quaternionic triple $I', J', K' \in \operatorname{Mat}(2)$. The ambiguity here is due to the ambiguity of a choice of Ω_I in a centralizer of ρ_I. The form Ω_I is defined up to a constant multiplier, because this centralizer is 1-dimensional.

The group $SL(2)$ acts transitively on the set of quaternionic triples. Consider $h \in SL(2)$ which maps I, J, K to $I', J', K' \in \mathrm{Mat}(2)$. Then $h(g_I)$ is proportional to $g_{I'}$.

Step 5: To finish the proof of Theorem 3.8 (ii), it remains to show that the $SL(2)$-invariant quadratic form g defining the trisymplectic structure $\mathbf{\Omega}$ is unique, up to a constant. Indeed, let g be such a form; then $g = \Omega(\cdot, I\cdot)$, for some $\Omega \in \mathbf{\Omega}$ and $I \in \mathrm{Mat}(2)$, satisfying $I^2 = -1$. Since g is $SL(2)$-invariant, the form Ω is ρ_I-invariant, hence g is proportional to the form g_I constructed above. We proved Theorem 3.8. ■

4 $SL(2)$-webs and trisymplectic structures

In this section we introduce the notion of trisymplectic structures on manifolds, study its reduction to quotients, and explain how they are related to holomorphic $SL(2)$-webs.

The trisymplectic structures and trisymplectic reduction were previously considered in a context of framed instanton bundles by Hauzer and Langer ([HL, Sections 7.1 and 7.2]). However, their approach is significantly different from ours, because they do not consider the associated $SL(2)$-web structures.

4.1 Trisymplectic structures on manifolds

Definition 4.1. A trisymplectic structure *on a complex manifold is a 3-dimensional space $\mathbf{\Omega} \subset \Omega^2 M$ of closed holomorphic 2-forms such that at any $x \in M$, the evaluation $\mathbf{\Omega}(x)$ gives a trisymplectic structure on the tangent space $T_x M$. A complex manifold equipped with a trisymplectic structure is called a* **trisymplectic manifold**.

Clearly, trisymplectic manifolds must have even complex dimension. Notice also that Theorem 3.8 implies the equivalence between the Definition above and Definition 1.1.

A similar notion is called a *hypersymplectic structure* by Hauzer and Langer in [HL, Definition 7.1]. A complex manifold (X, g) equipped with a non-degenerate holomorphic symmetric form is called **hypersymplectic** in [HL] if there are three complex structures I, J and K satisfying quaternionic relations and $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w)$. Clearly, one can then define three nondegerate symplectic forms $\omega_1(v, w) = g(Iv, w)$, $\omega_2(v, w) = g(Jv, w)$ and $\omega_3(v, w) = g(Kv, w)$ which generate a 3-dimensional subspace of holomorphic 2-forms $\mathbf{\Omega} \subset \Omega^2 M$. We require, in addition, that every nonzero, degenerate linear combination of ω_1, ω_2 and ω_3 has rank $\dim X/2$.

We prefer, however, to use the term trisymplectic to avoid confusion with the hypersymplectic structures known in differential geometry (see [AD, DS]), where a hypersymplectic structure is a 3-dimensional space W of differential 2-forms on a real manifold which satisfy the same rank assumptions as in Definition 4.1, and, in addition, contain a non-trivial degenerate 2-form (for complex-linear 2-forms, this last assumption is automatic).

Definition 4.2. *Let η be a $(p,0)$-form on a complex manifold M. Consider the set* Null_η *of all (1,0)-vectors $v \in T^{1,0}M$ satisfying $\eta \lrcorner v = 0$, where $\lrcorner$ denotes the contraction. Then* Null_η *is called* **the null-space***, or* **an annihilator***, of η.*

Lemma 4.3. *Let η be a closed $(p,0)$-form for which* Null_η *is a sub-bundle in $T^{1,0}(M)$. Then* Null_η *is holomorphic and involutive, that is, satisfies*

$$[\mathrm{Null}_\eta, \mathrm{Null}_\eta] \subset \mathrm{Null}_\eta\,.$$

Proof: The form η is closed and hence holomorphic, therefore Null_η is a holomorphic bundle. To prove that Null_η is involutive, we use the Cartan's formula, expressing de Rham differential in terms of commutators and Lie derivatives. Let $X \in T^{1,0}(M)$, $Y, Z \in \mathrm{Null}_\eta$. Then Cartan's formula gives $0 = d\eta(X,Y,Z) = \eta(X,[Y,Z])$. This implies that $[Y,Z]$ lies in Null_η. ∎

Lemma 4.3 can be used to construct holomorphic $SL(2)$-webs on manifolds, as follows.

Theorem 4.4. *Let M be a complex manifold, and $\mathbf{\Omega} \subset \Lambda^{2,0}(M)$ a 3-dimensional space of closed holomorphic (2,0)-forms. Assume that a generic form in $\mathbf{\Omega}$ is non-degenerate, and for each degenerate form $\Omega \in \mathbf{\Omega}$, the null-space* Null_η *is a sub-bundle of TM of dimension $\frac{1}{2}\dim M$. Then there is a holomorphic $SL(2)$-web (M, S_t), $t \in \mathbb{CP}^1$ on M such that each sub-bundle S_t is a null-space of a certain $\Omega_t \in \mathbf{\Omega}$.*

Proof: Theorem 4.4 follows immediately from Proposition 3.4 and Lemma 4.3. Indeed, at any point $x \in M$, the 3-dimensional space $\mathbf{\Omega}(x) \in \Lambda^{2,0}(T_xM)$ satisfies assumptions of Proposition 3.4, hence induces an action of the matrix algebra $H \cong \mathrm{Mat}(2)$ on T_xM. Denote by $Z \subset \mathbf{\Omega}$ the set of degenerate forms. From Proposition 3.4 (C) we obtain that the projectivization $\mathbb{P}Z \subset \mathbb{P}\mathbf{\Omega}$ is a non-singular quadric, isomorphic to $\mathbb{CP}^1$. For each $t \in \mathbb{Z}$, the corresponding zero-space $S_t \subset TM$ is a sub-bundle of dimension $\frac{1}{2}\dim M$, and for distinct t, the bundles S_t are obviously transversal. Also, Lemma 4.3 implies that the bundles S_t are involutive. Finally, the projection operators associated to S_t, S'_t generate a subalgebra isomorphic to $\mathrm{Mat}(2)$, as follows from Proposition 3.4. We have shown that $S_t, t \in \mathbb{P}Z \cong \mathbb{CP}^1$ is indeed an $SL(2)$-web. ∎

In particular, every trysimplectic manifold has an induced a holomorphic $SL(2)$-web.

Definition 4.5. *Let (M, S_t), $t \in \mathbb{CP}^1$ be a complex manifold equipped with a holomorphic $SL(2)$-web. Assume that there is a 3-dimensional space $\mathbf{\Omega} \subset \Lambda^{2,0}(M)$ of closed holomorphic 2-forms such that for each $t \in \mathbb{CP}^1$ there exists $\Omega_t \in \mathbf{\Omega}$ such that the $S_t = \mathrm{Null}_{\Omega_t}$. Then $\mathbf{\Omega}$ is called* **a trisymplectic structure generating the $SL(2)$-web** *$S_t, t \in \mathbb{CP}^1$.*

4.2 Chern connection on $SL(2)$-webs and trisymplectic structures

The following theorem is proven in the same way as one proves that the Kähler forms on a hyperkähler manifold are preserved by the Obata connection. Indeed, a trisymplectic structure is a complexification of a hyperkähler structure, and the Chern connection corresponds to a complexification of the Obata connection on a hyperkähler manifold.

Theorem 4.6. *Let $\mathbf{\Omega}$ be a trisymplectic structure generating an $SL(2)$-web on a complex manifold M. Denote by ∇ the corresponding Chern connection. Then $\nabla\Omega = 0$, for each $\omega \in \mathbf{\Omega}$.*

Proof: Let $(M, \mathbf{\Omega})$ be a trisymplectic manifold, and

$$\rho: \ \mathfrak{sl}(2) \longrightarrow \operatorname{End}(\Lambda^* M)$$

the corresponding multiplicative action of $\mathfrak{sl}(2)$ associated to $\mathfrak{g} \cong \mathfrak{sl}(2) \subset \operatorname{End}(TM)$ constructed in Proposition 3.4. By Proposition 3.4 (B), $\mathbf{\Omega}$ is an irreducible $\mathfrak{sl}(2)$-module. Choose a Cartan subalgebra in $\mathfrak{sl}(2)$, and let $\Omega^i(M) = \bigoplus_{p+q=i} \Omega^{p,q}(M)$ be the multiplicative weight decomposition associated with this Cartan subalgebra, with $\Omega^i(M) := \Lambda^{i,0}(M)$. We write the corresponding weight decomposition of $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = \mathbf{\Omega}^{2,0} \oplus \mathbf{\Omega}^{1,1} \oplus \mathbf{\Omega}^{0,2}.$$

Clearly

$$\Omega^i(M) = \bigoplus_{p+q=i} \Omega^{p,0}(M) \otimes \Omega^{0,q}(M), \tag{4.1}$$

since $\Omega^{p,0}(M) \otimes \Omega^{0,q}(M) = \Omega^{p,q}(M)$.

Consider the Chern connection as an operator

$$\Omega^i(M) \xrightarrow{\nabla} \Omega^i(M) \otimes \Omega^1(M)$$

(this makes sense, because ∇ is a holomorphic connection), and let

$$\Omega^{p,q}(M) \xrightarrow{\nabla^{1,0}} \Omega^{p,q}(M) \otimes \Omega^{1,0}(M), \quad \Omega^{p,q}(M) \xrightarrow{\nabla^{0,1}} \Omega^{p,q}(M) \otimes \Omega^{0,1}(M)$$

be its weight components. Since ∇ is torsion-free, one has

$$\partial\eta = \operatorname{Alt}(\nabla\eta), \tag{4.2}$$

where ∂ is the holomorphic de Rham differential, and

$$\operatorname{Alt}: \ \Omega^i(M) \otimes \Omega^1(M) \longrightarrow \Omega^{i+1}(M)$$

the exterior multiplication. Denote by $\Omega_{0,2}$, $\Omega_{2,0}$ generators of the 1-dimensional spaces $\mathbf{\Omega}^{2,0}, \mathbf{\Omega}^{0,2} \subset \mathbf{\Omega}$. Since $\partial\Omega_{2,0} = 0$, and the multiplication map $\Omega^{0,1}(M) \otimes$

$\Omega^{2,0}(M) \longrightarrow \Omega^3(M)$ is injective by (4.1), (4.2) implies that $\nabla^{0,1}(\Omega_{2,0}) = 0$. Similarly, $\nabla^{1,0}(\Omega_{0,1}) = 0$. However, since $\mathbf{\Omega}$ is irreducible as a representation of $\mathfrak{sl}(2)$, there exist an expression of form $\Omega_{2,0} = g(\Omega_{0,2})$, where $g \in U_{\mathfrak{g}}$ is a polynomial in $\mathfrak{g}$. Since the Chern connection ∇ commutes with g, this implies that

$$0 = g(\nabla^{1,0}\Omega_{0,2}) = \nabla^{1,0}(g\Omega_{0,2}) = \nabla^{1,0}\Omega_{2,0}.$$

We have proved that both weight components of $\nabla\Omega_{2,0}$ vanish, thus $\nabla\Omega_{2,0} = 0$. Acting on $\Omega_{2,0}$ by $\mathfrak{sl}(2)$ again, we obtain that $\nabla\Omega = 0$ for all $\Omega \in \mathbf{\Omega}$. ■

4.3 Trisymplectic reduction

Definition 4.7. *Let G be a compact Lie group acting on a complex manifold equipped with a trisymplectic structure $\mathbf{\Omega}$ generating an $SL(2)$-web. Assume that G preserves $\mathbf{\Omega}$.* **A trisymplectic moment map** $\boldsymbol{\mu}_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbf{\Omega}^*$ *takes vectors $\Omega \in \mathbf{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$ and maps them to a holomorphic function $f \in \mathcal{O}_M$, such that $df = \Omega \lrcorner g$, where $\Omega \lrcorner g$ denotes the contraction of Ω and the vector field g. A moment map is called* **equivariant** *if it equivariant with respect to coadjoint action of G on $\mathfrak{g}^*$ Further on, we shall always assume that all moment maps we consider are equivariant.*

Since $d\Omega = 0$, and $\operatorname{Lie}_g \Omega = 0$, Cartan's formula gives $0 = \operatorname{Lie}_g(\Omega) = d(\Omega \lrcorner g)$, hence the contraction $\Omega \lrcorner g$ is closed. Therefore, existence of the moment map is equivalent to exactness of this closed 1-form for each $\Omega \in \mathbf{\Omega}, g \in \mathfrak{g} = \operatorname{Lie}(G)$. Therefore, the existence of a moment map is assured whenever M is simply connected. The existence of an *equivariant* moment map is less immediate, and depends on certain cohomological properties of G (see e.g. [HKLR]).

Definition 4.8. *Let $(M, \mathbf{\Omega}, S_t)$ be a trisymplectic structure on a complex manifold M. Assume that M is equipped with an action of a compact Lie group G preserving $\mathbf{\Omega}$, and an equivariant trisymplectic moment map*

$$\boldsymbol{\mu}_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbf{\Omega}^*.$$

Consider a G-invariant vector $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbf{\Omega}^*$ (usually, one sets $c = 0$), and let $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$ be the corresponding **level set** *of the moment map. Consider the action of the corresponding complex Lie group $G_{\mathbb{C}}$ on $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Assume that it is proper and free. Then the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is a smooth manifold called* **the trisymplectic quotient** *of $(M, \mathbf{\Omega}, S_t)$, denoted by $M /\!\!/\!\!/\!\!/ G$.*

As we shall see, the trisymplectic quotient is related to the usual hyperkähler quotient in the same way as the hyperkähler quotient (denoted by $/\!\!/\!\!/$) is related to the symplectic quotient, denoted by $/\!\!/$. In heuristic terms, the hyperkähler quotient can be considered as a "complexification" of a symplectic quotient; similarly, the trisymplectic quotient is a "complexification" of a hyperkähler quotient.

The non-degeneracy condition of Theorem 4.9 below is necessary for the trisymplectic reduction process, in the same way as one would need some non-degeneracy if one tries to perform the symplectic reduction on a pseudo-Kähler manifold. On a Kähler (or a hyperkähler) manifold it is automatic because the metric is positive definite, but otherwise it is easy to obtain counterexamples (even in the simplest cases, such as S^1-action on $\mathbb{C}^2$ with an appropriate pseudo-Kähler metric).

Theorem 4.9. *Let $(M, \mathbf{\Omega})$ be a trisymplectic manifold. Assume that M is equipped with an action of a compact Lie group G preserving $\mathbf{\Omega}$ and a trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbf{\Omega}^*$. Assume, moreover, that the image of $\mathfrak{g} = \operatorname{Lie}(G)$ in TM is non-degenerate at any point (in the sense of Definition 3.6). Then the trisymplectic quotient $M /\!\!/\!\!/\!\!/ G := \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/G_{\mathbb{C}}$ is naturally equipped with a trisymplectic structure.*[1]

For a real version of this theorem, please see [DS].

The proof of Theorem 4.9 takes the rest of this section. First, we shall use the following easy definition and an observation.

Definition 4.10. *Let $B \subset TM$ be an involutive sub-bundle in a tangent bundle to a smooth manifold M. A form $\eta \in \Lambda^i M$ is called* **basic with respect to** *B if for any $X \in B$, one has $\eta \lrcorner X = 0$ and $\operatorname{Lie}_X \eta = 0$.*

The following claim is clear.

Claim 4.11. *Let $B \subset TM$ be an involutive sub-bundle in a tangent bundle to a smooth manifold M. Consider the projection $M \xrightarrow{\pi} M'$ onto its leaf space, which is assumed to be Hausdorff. Let $\eta \in \Lambda^i M$ be a basic form on M. Then $\eta = \pi^*\eta'$, for an appropriate form η' on M'.* ■

Return to the proof of Theorem 4.9. Let I, J, K be a quaternionic basis in $\operatorname{Mat}(2)$, $\Omega_I \in \mathbf{\Omega}$ a ρ_I-invariant form chosen as in Theorem 3.8 (ii), and $g := \Omega_I(\cdot, I\cdot)$ the corresponding non-degenerate, complex linear symmetric form on M. By its construction, g is holomorphic, and by Theorem 3.8 (ii), $SL(2)$-invariant. Let $N \subset M$ be a level set of the moment map $N := \boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$. Choose a point $m \in N$, and let $\mathfrak{g}_m \subset T_m M$ be the image of $\mathfrak{g} = \operatorname{Lie} G$ in $T_m M$. Then, for each $v \in \mathfrak{g}_m$, one has

$$d\mu_I(v, \cdot) = \Omega_I(v, \cdot), \tag{4.3}$$

where $\mu_I : M \to \mathfrak{g}^*$ is the holomorphic moment map associated with the symplectic form Ω_I.

On the other hand, $\Omega_I(v, \cdot) = -g(Iv, \cdot)$. Therefore, $T_m N \subset T_m M$ is an orthogonal complement (with respect to g) to the space $\langle I\mathfrak{g}_m, J\mathfrak{g}_m, K\mathfrak{g}_m\rangle$ generated by $I(\mathfrak{g}_m), J(\mathfrak{g}_m), K(\mathfrak{g}_m)$:

$$T_m N = \langle I\mathfrak{g}_m, J\mathfrak{g}_m, K\mathfrak{g}_m\rangle_g^{\perp}. \tag{4.4}$$

[1] In the statement of Theorem 4.9 we implicitly assume that the quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$ is well-defined, which is not always the case.

By (4.3), for any $v \in \mathfrak{g}_m$, and $w \in T_m N$, one has $\Omega_I(v, w) = 0$. Also, G preserves all forms from $\mathbf{\Omega}$, hence $\operatorname{Lie}_v \Omega_i = 0$. Therefore, Ω_I is basic with respect to the distribution $V \subset TN$ generated by the image of Lie algebra $\mathfrak{g} \longrightarrow TN$.

Consider the quotient map $N \xrightarrow{\pi} N/G_{\mathbb{C}} = M'$. To prove that M' is a trisymplectic manifold, we use Claim 4.11, obtaining a 3-dimensional space of holomorphic 2-forms $\mathbf{\Omega}' \subset \Lambda^{2,0}(M')$, with $\mathbf{\Omega}\big|_N = \pi^* \mathbf{\Omega}'$. To check that $\mathbf{\Omega}'$ is a trisymplectic structure, it remains only to establish the rank conditions.

Let $W \subset T_m N$ be a subspace complementary to $\mathfrak{g}_m \subset T_m N$. Clearly, for any $\Omega \in \mathbf{\Omega}$, the rank of the corresponding form $\Omega' \in \mathbf{\Omega}'$ at the point $m' = \pi(m)$ is equal to the rank of $\Omega\big|_W$.

Let $W_1 \subset T_m M$ be a subspace obtained as $H \cdot \mathfrak{g}_m$, where $H \cong \operatorname{Mat}(2) \subset \operatorname{End}(T_m M)$ is the standard action of the matrix algebra defined as in Subsection 3.1. By the non-degeneracy assumption of Theorem 4.9, the restriction $g\Big|_{W_1}$ is non-degenerate, hence the orthogonal complement $W_1^\perp$ satisfies $T_m M = W_1 \oplus W_1^\perp$. From (4.4) we obtain $W_1^\perp \subset T_m N$, with $W_1^\perp \oplus \mathfrak{g}_m = T_m N$. Therefore, $W := W_1^\perp$ is complementary to $\mathfrak{g}_m$ in $T_m N$. The space $(W, \mathbf{\Omega}\big|_W)$ is trisymplectic, as follows from Claim 3.5. Therefore, the forms $\mathbf{\Omega}' \subset \Lambda^{2,0}(M')$ define a trisymplectic structure on M'. We have proved Theorem 4.9. ∎

5 Trihyperkähler reduction

5.1 Hyperkähler reduction

Let us start by recalling some well-known definitions.

Definition 5.1. *Let G be a compact Lie group acting on a hyperkähler manifold M by hyperkähler isometries.* **A hyperkähler moment map** *is a smooth map $\mu : M \to \mathfrak{g} \otimes \mathbb{R}^3$ such that:*

(1) *μ is G-equivariant, i.e. $\mu(g \cdot m) = \operatorname{Ad}^*_{g^{-1}} \mu(m)$;*

(2) *$\langle d\mu_i(v), \xi \rangle = \omega_i(\xi^*, v)$, for every $v \in TM$, $\xi \in \mathfrak{g}$ and $i = 1, 2, 3$, where μ_i denotes one of the 3 components of μ, ω_i is one the the Kähler forms associated with the hyperkähler structure, and ξ^* is the vector field generated by ξ.*

Definition 5.2. *Let $\xi_i \in \mathfrak{g}^*$ $(i = 1, 2, 3)$ be such that $\operatorname{Ad}^*_g \xi_i = \xi_i$, so that G acts on $\mu^{-1}(\xi_1, \xi_2, \xi_3)$; suppose that this action is free. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called the* **hyperkähler quotient** *of M.*

Theorem 5.3. *Let M be a hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms, and admitting a hyperkähler moment map. Then the hyperkähler quotient $M /\!\!/\!\!/ G$ is equipped with a natural hyperkähler structure.*

Proof: See [HKLR], [Nak, Theorem 3.35] ∎

5.2 Trisymplectic reduction on the space of twistor sections

Let M be a hyperkähler manifold, $L \in \mathbb{CP}^1$ an induced complex structure and $\mathsf{ev}_L : \operatorname{Sec}(M) \longrightarrow (M, L)$ the corresponding evaluation map, mapping a section $s : \mathbb{CP}^1 \longrightarrow \operatorname{Tw}(M)$ to $s(L) \in (M, L) \subset \operatorname{Tw}(M)$. Consider the holomorphic form $\Omega_L \in \Lambda^{2,0}(M, L)$ constructed from a hyperkähler structure as in (2.2). Denote by $\mathbf{\Omega}$ the space of holomorphic forms on $\operatorname{Sec}(M)$ generated by $\mathsf{ev}_L^*(\Omega_L)$ for all $L \in \mathbb{CP}^1$.

Claim 5.4. *$\mathbf{\Omega}$ is a trisymplectic structure on the space $\operatorname{Sec}_0(M)$ of regular sections, generating the standard $SL(2)$-web.*

Proof: Consider the bundle $\mathcal{O}(2)$ on $\mathbb{CP}^1$, and let $\pi^*\mathcal{O}(2)$ be its lift to the twistor space $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{CP}^1$. Denote by $\Omega^2_\pi \operatorname{Tw}(M)$ the sheaf of fiberwise 2-forms on $\operatorname{Tw}(M)$. The bundle $\Omega^2_\pi \operatorname{Tw}(M)$ can be obtained as a quotient

$$\Omega^2_\pi \operatorname{Tw}(M) := \frac{\Omega^2 \operatorname{Tw}(M)}{\pi^*\Omega^1\mathbb{CP}^1 \wedge \Omega^1 \operatorname{Tw}(M)}.$$

It is well known (see e.g. [HKLR]), that the fiberwise symplectic structure depends on $t \in \mathbb{CP}^1$ holomorphically, and, moreover, $\operatorname{Tw}(M)$ is equipped with a holomorphic 2-form $\Omega_{tw} \in \pi^*\mathcal{O}(2) \otimes \Omega^2_\pi \operatorname{Tw}(M)$ inducing the usual holomorphic symplectic forms on the fibers, see [HKLR, Theorem 3.3(iii)].

Given $S \in \operatorname{Sec}(M)$, the tangent space $T_s\operatorname{Sec}(M)$ is identified with the space of global sections of a bundle $T_\pi \operatorname{Tw}(M)\big|_S$. Therefore, any vertical 2-form $\Omega_1 \in \Omega^2_\pi \operatorname{Tw}(M) \otimes \pi^*\mathcal{O}(i)$ defines a holomorphic 2-form on $\operatorname{Sec}_0(M)$ with values in the space of global sections $\Gamma(\mathbb{CP}^1, \mathcal{O}(i))$.

Denote by A the space $\Gamma(\mathbb{CP}^1, \mathcal{O}(2))$. A fiberwise holomorphic $\mathcal{O}(2)$-valued 2-form gives a 2-form on NS, for each $S \in \operatorname{Sec}(M)$, with values in A. Therefore, for each $\alpha \in A^*$, one obtains a 2-form $\Omega_{tw}(\alpha)$ on $\operatorname{Sec}(M)$ as explained above. Let $\mathbf{\Omega}$ be a 3-dimensional space generated by $\Omega_{tw}(\alpha)$ for all $\alpha \in A^*$.

Consider a map $\epsilon_L : A \longrightarrow \mathcal{O}(2)\big|_L \cong \mathbb{C}$ evaluating $\gamma \in \Gamma(\mathcal{O}(2))$ at a point $L \in \mathbb{CP}^1$. By definition, the 2-form $\Omega_{tw}(\epsilon_L)$ is proportional to $\mathsf{ev}_L^* \Omega_L$. Therefore, $\mathbf{\Omega}$ contains $\mathsf{ev}_L^* \Omega_L$ for all $L \in \mathbb{CP}^1$.

Counting parameters, we obtain that any element $x \in A^*$ is a sum of two evaluation maps: $x = a\epsilon_{L_1} + b\epsilon_{L_2}$. When $a, b \neq 0$ and L_1, L_2 are distinct, the corresponding 2-form $a\,\mathsf{ev}_{L_1}^* \Omega_{L_1} + b\,\mathsf{ev}_{L_2}^* \Omega_{L_2}$ is clearly non-degenerate on $\operatorname{Sec}_0(M)$. Indeed, the map

$$\operatorname{Sec}_0(M) \xrightarrow{\mathsf{ev}_{L_1} \times \mathsf{ev}_{L_2}} (M, L_1) \times (M, L_2)$$

is etale, and any linear combination $a\Omega_{L_1} + b\Omega_{L_2}$ with non-zero a, b is nondegenerate on $(M, L_1) \times (M, L_2)$. When a or b vanish, the corresponding form (if non-zero) is proportional to $\mathsf{ev}_{L_i}^* \Omega_{L_i}$, hence its rank is $\dim M = \frac{1}{2} \dim \operatorname{Sec}(M)$.

We have shown that $\mathbf{\Omega}$ is a trisymplectic structure. Clearly, the annihilators of $\mathsf{ev}_L^*(\Omega_L)$ form the standard 3-web on $\operatorname{Sec}(M)$. Therefore, the trisymplectic

structure $\mathbf{\Omega}$ generates the standard $SL(2)$-web, described in Section 2.3 above. ■

Now let G be a compact Lie group acting on M by hyperkähler isometries; assume that the hyperkähler moment map for the action of G on M exists. Let $\operatorname{Sec}_0(M)$ be the space of regular twistor sections, considered with the induced $SL(2)$-web and trisymplectic structure. The previous Claim immediately implies the following Proposition.

Proposition 5.5. *Given any $L \in \mathbb{CP}^1$, let $\mu_L : (M, L) \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ denote the corresponding holomorphic moment map, and consider the composition*

$$\boldsymbol{\mu}_L := \mu_L \circ \mathsf{ev}_L : \operatorname{Sec}(M) \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}.$$

Then

$$\boldsymbol{\mu}_{\mathbb{C}} := \boldsymbol{\mu}_I \oplus \boldsymbol{\mu}_J \oplus \boldsymbol{\mu}_K : \operatorname{Sec}_0(M) \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}^3$$

is a trisymplectic moment map on $\operatorname{Sec}_0(M)$, for an appropriate identification $\mathbb{C}^3 \cong \mathbf{\Omega}$.

Proof: Clearly, $\boldsymbol{\mu}_L$ is a moment map for the action of G on $\operatorname{Sec}_0(M)$ associated with a degenerate holomorphic 2-form $\mathsf{ev}_L^*(\Omega_L)$. Indeed, for any $g \in \mathfrak{g} = \operatorname{Lie}(G)$, one has $d\mu_L(G) = \Omega_L \lrcorner\, g$, because μ_L is a moment map form G acting on (M, L). Then $d\boldsymbol{\mu}_L(g) = (\mathsf{ev}_L^* \Omega_L) \lrcorner\, g$.

However, by Claim 5.4, $\mathbf{\Omega} = \mathsf{ev}_I^* \Omega_I \oplus \mathsf{ev}_J^* \Omega_J \oplus \mathsf{ev}_K^* \Omega_K$, hence the moment map $\boldsymbol{\mu}$ associated with $\mathbf{\Omega}$ is expressed as an appropriate linear combination of $\boldsymbol{\mu}_I, \boldsymbol{\mu}_J, \boldsymbol{\mu}_K$. ■

5.3 Trihyperkähler reduction on the space of twistor sections

Let $\operatorname{Tw}(M) = M \times \mathbb{CP}^1$ be the twistor space of the hyperkähler manifold M, considered as a Riemannian manifold with its product metric. We normalize the Fubini-Study metric on the second component of $\operatorname{Tw}(M) = M \times \mathbb{CP}^1$ in such a way that $\int_{\mathbb{CP}^1} \operatorname{Vol}_{\mathbb{CP}^1}$ of the Riemannian volume form is 1.

Claim 5.6. *Let ϕ be the area function $\operatorname{Sec}(M) \xrightarrow{\phi} \mathbb{R}^{>0}$ mapping a curve $S \in \operatorname{Sec}(M)$ to its its Riemannian volume $\int_S \operatorname{Vol}_S$. Then ϕ is a Kähler potential, that is, $dd^c\phi$ is a Kähler form on $\operatorname{Sec}(M)$, where d^c is the usual twisted differential, $d^c := -IdI$.*

Proof: See [KV, Proposition 8.15]. ■

Claim 5.6 leads to the following Proposition.

Proposition 5.7. *Assume that G is a compact Lie group acting on M by hyperkähler automorphisms, and admitting a hyperkähler moment map. Consider the corresponding action of G on $\operatorname{Sec}_0(M)$, and let $\omega_{\mathrm{Sec}} = dd^c\phi$ be the Kähler*

form on $\operatorname{Sec}_0(M)$ *constructed in Claim 5.6. Then the corresponding moment map can be written as*

$$\boldsymbol{\mu}_{\mathbb{R}}(x) := \operatorname{Av}_{L\in\mathbb{CP}^1} \mu_L^{\mathbb{R}}(x),$$

where $\operatorname{Av}_{L\in\mathbb{CP}^1}$ *denotes the operation of taking average over* $\mathbb{CP}^1$*, and* $\mu_L^{\mathbb{R}} : (M,L) \longrightarrow \mathfrak{g}^*$ *is the Kähler moment map associated with an action of* G *on* (M,L)*.*

Proof: Let (X, I, ω) be a Kähler manifold, ϕ a Kähler potential on X, and G a real Lie group preserving ϕ and acting on X holomorphically. Then an equivariant moment map can be written as

$$\mu(g) = -\operatorname{Lie}_{I(g)} \phi, \tag{5.1}$$

where $g \in \operatorname{Lie}(G)$ is an element of the Lie algebra. Indeed, $\omega = dd^c\phi$, hence

$$\operatorname{Lie}_{I(g)} \phi = d\phi \lrcorner (I(g)) = (d^c\phi) \lrcorner g,$$

where $\lrcorner$ denotes a contraction of a differential form with a vector field, and

$$d \operatorname{Lie}_{I(g)} \phi = d((d^c\phi) \lrcorner g) = \operatorname{Lie}_g(d^c\phi) - (dd^c\phi) \lrcorner g = -\omega \lrcorner g$$

by Cartan's formula. Applying this argument to $X = \operatorname{Sec}(M)$ and $\phi = \operatorname{Area}(S)$, we obtain that $\boldsymbol{\mu}_{\mathbb{R}}(S)(g)$ is a Lie derivative of ϕ along $I(g)$.

To prove that $\boldsymbol{\mu}_{\mathbb{R}}(g)$ is equal to an average of the moment maps $\mu_L^{\mathbb{R}}(g)$, we notice that (as follows from [KV], (8.12) and Lemma 4.4), for any fiberwise tangent vectors $x, y \in T_\pi \operatorname{Tw}(M)$, one has

$$dd^c\phi(x, Iy) = \int_S (x,y)_H \operatorname{Vol}_{\mathbb{CP}^1},$$

where $\operatorname{Vol}_{\mathbb{CP}^1}$ is the appropriately normalized volume form, and $(\cdot,\cdot)_H$ the standard Riemannian metric on $\operatorname{Tw}(M) = M \times S^2$. Taking $g = y$, we obtain

$$d(\boldsymbol{\mu}_{\mathbb{R}} g)(x) = \int_S (x,g)_H \operatorname{Vol}_{\mathbb{CP}^1} = \int_S d(\mu_L^{\mathbb{R}} g)(x) \operatorname{Vol}_{\mathbb{CP}^1}.$$

The last formula is a derivative of an average of $d\mu_L^{\mathbb{R}}(g)$ over $L \in \mathbb{CP}^1$. ■

From Proposition 5.7 it is is apparent that a trisymplectic quotient of the space $\operatorname{Sec}_0(M)$ can be obtained using the symplectic reduction associated with the real moment map $\boldsymbol{\mu}_{\mathbb{R}}$. This procedure is called **the trihyperkähler reduction** of $\operatorname{Sec}_0(M)$.

Definition 5.8. *The map* $\boldsymbol{\mu} := \boldsymbol{\mu}_{\mathbb{R}} \oplus \boldsymbol{\mu}_{\mathbb{C}} : \operatorname{Sec}_0(M) \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^7$*, where* $\boldsymbol{\mu}_{\mathbb{C}}$ *is the trisymplectic moment map constructed in Proposition 5.5 and* $\boldsymbol{\mu}_{\mathbb{R}}$ *is the Kähler moment map constructed in Proposition 5.7, is called* **the trihyperkähler moment map** *on* $\operatorname{Sec}_0(M)$*.*

Definition 5.9. *Let $c \in \mathfrak{g}^* \otimes \mathbb{R}^7$ be a G-invariant vector. Consider the space* $\mathrm{Sec}_0(M)$ *of the regular twistor sections. Then the quotient* $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G := \boldsymbol{\mu}^{-1}(c)/G$ *is called* **the trihyperkähler reduction** *of* $\mathrm{Sec}_0(M)$.

Remark 5.10. *Note that the trihyperkähler reduction $\boldsymbol{\mu}^{-1}(c)/G$ of* $\mathrm{Sec}_0(M)$ *coincides with the trisymplectic quotient $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$, provided this last quotient is well-defined, i.e. all $G_{\mathbb{C}}$-orbits within $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(c)$ are GIT-stable with respect to a suitable linearization of the action. Indeed, $(\mu_{\mathbb{C}} \oplus \mu_{\mathbb{R}})^{-1}(c)/G$ is precisely the space of stable $G_{\mathbb{C}}$-orbits in $\mu_{\mathbb{C}}^{-1}(c)$.*

It follows from Theorem 4.9 that $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ is equipped with an $SL(2)$-web generated by a natural trisymplectic structure $\boldsymbol{\Omega}$, provided that the image of $\mathfrak{g} = \mathrm{Lie}(G)$ in TM is non-degenerate at any point, in the sense of Definition 3.6.

We are finally ready to state the main result of this paper

Theorem 5.11. *Let M be flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient $M/\!\!/\!\!/G$ is smooth. Then there exists an open embedding* $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G \xrightarrow{\Psi} \mathrm{Sec}_0(M/\!\!/\!\!/G)$*, which is compatible with the trisymplectic structures on* $\mathrm{Sec}_0(M)/\!\!/\!\!/\!\!/G$ *and* $\mathrm{Sec}_0(M/\!\!/\!\!/G)$.

In particular, it follows that if M is a flat hyperkähler manifold, then the trihyperkähler reduction of $\mathrm{Sec}_0(M)$ is a smooth trisymplectic manifold whose dimension is twice that of the hyperkähler quotient of M.

The flatness condition is mostly a technical one, but it will suffice for our main goal, which is a description of the moduli space of instanton bundles on $\mathbb{CP}^3$ (see Section 8 below).

We do believe that the conclusions of Theorem 5.11 should hold without such a condition. The crucial point is Proposition 6.1 below, for which we could not find a proof without assuming flatness; other parts of our proof, which will be completed at the end of Subsection 7.2, do work without it.

6 Moment map on twistor sections

In this Section, we let M be a *flat* hyperkähler manifold. More precisely, let M be an open subset of a quaternionic vector space V, equipped with a flat metric; completeness of the metric is not relevant. Thus $\mathrm{Tw}(M)$ is isomorphic to the corresponding open subset of $\mathrm{Tw}(V) = V \otimes \mathcal{O}_{\mathbb{CP}^1}(1)$, and $\mathrm{Sec}_0(M) = \mathrm{Sec}(M)$ is the open subset of $\mathrm{Sec}(V) = V \otimes_{\mathbb{C}} \Gamma(\mathcal{O}_{\mathbb{CP}^1}(1)) \simeq V \otimes_{\mathbb{R}} \mathbb{C}^2$ consisting of those sections of $V \otimes \mathcal{O}_{\mathbb{CP}^1}(1)$ that take values in $M \subset V$.

More precisely, let $[z : w]$ be a choice of homogeneous coordinates on $\mathbb{CP}^1$, so that $\Gamma(\mathcal{O}_{\mathbb{CP}^1}(1)) \simeq \mathbb{C}z \oplus \mathbb{C}w$. A section $\sigma \in Sec_0(M)$ will of the the form $\sigma(z, w) = zX_1 + wX_2$ such that $\sigma(z, w) \in M$ for every $[z : w] \in \mathbb{CP}^1$.

Let G be a compact Lie group acting on M by hyperkähler automorphisms, with $\mu : M \to \mathfrak{g}^* \otimes \langle I, J, K\rangle$ being the corresponding hyperkähler moment map;

let $\mu_I^{\mathbb{R}}$, $\mu_J^{\mathbb{R}}$, $\mu_K^{\mathbb{R}}$ denote its components. By definition, these components are the real moment maps associated with the symplectic forms $\omega_I, \omega_J, \omega_K$, respectively. Given a complex structure $L = aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$, we denote by $\mu_L^{\mathbb{R}}$ the corresponding real moment map,

$$\mu_L^{\mathbb{R}} = a\mu_I^{\mathbb{R}} + b\mu_J^{\mathbb{R}} + c\mu_K^{\mathbb{R}}. \tag{6.1}$$

The components μ_I, μ_J, μ_K of the hyperkähler moment map can be regarded as real-valued, quadratic polynomials on V. The corresponding complex linear polynomial functions $\boldsymbol{\mu}_I, \boldsymbol{\mu}_J, \boldsymbol{\mu}_K$ generate the trisymplectic moment map for $\mathrm{Sec}_0(M)$. Consider the decomposition $V \otimes_{\mathbb{R}} \mathbb{C}^2 = V_I^{1,0} \oplus V_I^{0,1}$, where $I \in \mathrm{End}\, V$ acts on $V_I^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ as $\sqrt{-1}$ and on $V_I^{0,1}$ as $-\sqrt{-1}$. We may regard the trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}} : \mathrm{Sec}_0(M) \longrightarrow \mathfrak{g}^* \otimes \mathbb{C}^3$ as a quadratic form Q on $\mathrm{Sec}_0(M) \simeq V_I^{1,0} \oplus V_I^{0,1}$, and express it as a sum of three components,

$$Q^{2,0} :\ V_I^{1,0} \otimes V_I^{1,0} \longrightarrow \mathfrak{g}^* \otimes \mathbb{C}, \qquad Q^{1,1} :\ V_I^{1,0} \otimes V_I^{0,1} \longrightarrow \mathfrak{g}^* \otimes \mathbb{C}$$

$$\text{and}\ \ Q^{0,2} : V_I^{0,1} \otimes V_I^{0,1} \longrightarrow \mathfrak{g}^* \otimes \mathbb{C}.$$

For each $L \in \mathbb{CP}^1$, let $\mu_L^{\mathbb{R}}$ be the real moment map, depending on $L \in \mathbb{CP}^1$ as in (6.1), and consider the evaluation map $\mathrm{Sec}_0(M) \xrightarrow{\mathsf{ev}_L} (M, L)$ (see Claim 5.4 for definition). Let also $\boldsymbol{\mu}_L^{\mathbb{R}} := \mathsf{ev}_L^* \mu_L^{\mathbb{R}}$ be the pullback of $\mu_L^{\mathbb{R}}$ to $\mathrm{Sec}_0(M)$.

From Proposition 5.5, the following description of the moment maps on $\mathrm{Sec}(M)$ can be obtained. This result will be used later on in the proof of Theorem 5.11.

Proposition 6.1. *Let G be a real Lie group acting on a flat hyperkähler manifold M by hyperkähler isometries, $\mathrm{Sec}(M) \xrightarrow{\boldsymbol{\mu}_{\mathbb{C}}} \mathfrak{g}^* \otimes \mathbb{C}^3$ the corresponding trisymplectic moment map. We consider the real moment map $\mu_L^{\mathbb{R}}$ as a $\mathfrak{g}^*$-valued function on $\mathrm{Tw}(M) = M \times \mathbb{CP}^1$. Let $S \in \mathrm{Sec}(M)$ be a point which satisfies $\boldsymbol{\mu}_{\mathbb{C}}(S) = 0$. Then $\mu_L^{\mathbb{R}}\big|_S$ is constant.*

Proof. We must show that for each $S \in \mathrm{Sec}(M)$ satisfying $\boldsymbol{\mu}_{\mathbb{C}}(S) = 0$, one has $\frac{d}{dL}\boldsymbol{\mu}_L^{\mathbb{R}}(S) = 0$.

We express $S \in V \otimes_{\mathbb{R}} \mathbb{C}$ as $S = s_L^{1,0} + s_L^{0,1}$, with $s_L^{1,0} \in V_L^{1,0}$ and $s_L^{0,1} \in V_L^{0,1}$ Then

$$\boldsymbol{\mu}_L^{\mathbb{R}}(S) = Q_L^{1,1}(s_L^{1,0}, \overline{s_L^{1,0}}) \tag{6.2}$$

where $Q_L^{1,1}$ denotes the (1,1)-component of $\boldsymbol{\mu}^{\mathbb{C}}$ taken with respect to L. This clear, because $Q^{1,1}$ is obtained by complexifying $\mu_L^{\mathbb{R}}$ (this is an L-invariant part of the hyperkähler moment map).

For an ease of differentiation, we rewrite (6.2) as

$$\boldsymbol{\mu}_L^{\mathbb{R}}(S) = Q(s_L^{1,0}, \overline{s_L^{1,0}}) = \mathrm{Re}(Q(s_L^{1,0}, s_L^{1,0})).$$

This is possible, because $s_L^{1,0} \in V_L^{1,0}$ and $\overline{s_L^{1,0}} \in V_L^{0,1}$, hence $Q_L^{1,1}$ is the only component of Q which is non-trivial on $(s_L^{1,0}, \overline{s_L^{1,0}})$. Then

$$\frac{d}{dL}\boldsymbol{\mu}_L^{\mathbb{R}}(S)\big|_{L=I} = \operatorname{Re}\left[Q\left(s_I^{1,0}, \frac{ds_L^{1,0}}{dL}\big|_{L=I}\right)\right]. \tag{6.3}$$

However, $\frac{ds_L^{1,0}}{dL}\big|_{L=I}$ is clearly proportional to $s_I^{0,1}$ (the coefficient of proportionality depends on the choice of parametrization on $\mathbb{CP}^1 \ni L$), hence (6.3) gives

$$\frac{d}{dL}\boldsymbol{\mu}_L^{\mathbb{R}}(S)\big|_{L=I} = \lambda \operatorname{Re}\left[Q(s_I^{1,0}, s_I^{0,1})\right]$$

and this quantity vanishes, because

$$Q(s_L^{1,0}, s_L^{0,1}) = Q^{1,1}(S) = \mu_L^{\mathbb{C}}(S).$$

■

7 Trisymplectic reduction and hyperkähler reduction

7.1 The tautological map $\tau : \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}(M/\!\!/\!\!/G)$

Let M be a hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler isometries, and admitting a hyperkähler moment map. A point in $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$ is represented by a section $S \in \operatorname{Sec}_0(M)$ which satisfies $\boldsymbol{\mu}_{\mathbb{C}}(S) = 0$ and $\boldsymbol{\mu}_{\mathbb{R}}(S) = 0$. The first condition, by Proposition 5.5, implies that for each $L \in \mathbb{CP}^1$, the corresponding point $S(L) \in (M, L)$ belongs to the zero set of the holomorphic symplectic map $\mu_L^{\mathbb{C}} : (M, L) \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$. Using the evaluation map defined in Claim 5.4, this is written as

$$\mu_L^{\mathbb{C}}(\mathsf{ev}_L(S)) = 0.$$

By Proposition 6.1, the real moment map $\mu_L^{\mathbb{R}}$ is constant on S:

$$\mu_L^{\mathbb{C}}(\mathsf{ev}_L(S)) = const.$$

By Proposition 5.7, the real part of the trihyperkähler moment map $\boldsymbol{\mu}_{\mathbb{R}}(S)$ is an averace of $\mu_L^{\mathbb{C}}(\mathsf{ev}_L(S))$ taken over all $L \in \mathbb{CP}^1$. Therefore, $\boldsymbol{\mu}_{\mathbb{C}}(S) = 0$ implies

$$\boldsymbol{\mu}_{\mathbb{R}}(S) = 0 \Leftrightarrow \mu_L^{\mathbb{C}}(\mathsf{ev}_L(S)) = 0 \ \ \forall L \in \mathbb{CP}^1.$$

We obtain that for each $S \in \operatorname{Sec}_0(M)$ which satisfies $\boldsymbol{\mu}_{\mathbb{R}}(S) = 0, \boldsymbol{\mu}_{\mathbb{C}}(S) = 0$, and each $l \in \mathbb{CP}^1$, one has

$$\mu_L^{\mathbb{C}}(x) = 0, \mu_L^{\mathbb{R}}(x) = 0, \tag{7.1}$$

where $x = \mathsf{ev}_L(S)$. A point $x \in (M, L)$ satisfying (7.1) belongs to the zero set of the hyperkähler moment map $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$. Taking a quotient over G, we obtain a map $S/G : \mathbb{CP}^1 \longrightarrow \mathrm{Tw}(M/\!\!/\!\!/G)$, because $M/\!\!/\!\!/G$ is a quotient of $\mu^{-1}(0)$ by G. This gives a map $\tau : \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}(M/\!\!/\!\!/G)$ which is called **a tautological map**. Note that $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$ has a trisymplectic structure by Theorem 4.9, outside of the set of its degenerate points (in the sense of Definition 3.6), and $\operatorname{Sec}_0(M/\!\!/\!\!/G)$ is a trisymplectic manifold by Proposition 5.5.

Proposition 7.1. *Let M be a flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler isometries, and admitting a hyperkähler moment map. Consider the tautological map*

$$\tau : \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}(M/\!\!/\!\!/G) \tag{7.2}$$

defined above. Then $\tau(\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G)$ belongs to the set $\operatorname{Sec}_0(M/\!\!/\!\!/G)$ of regular twistor sections in $\mathrm{Tw}(M/\!\!/\!\!/G)$. Moreover, the image of $\mathfrak{g}$ is non-degenerate, in the sense of Definition 3.6, and τ is a local diffeomorphism, compatible with the trisymplectic structure.

Proof. Step 0: We prove that for all points $S \in \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$, the image of $\mathfrak{g}$ is non-degenerate, in the sense of Definition 3.6. This is the only step of the proof where the flatness assumption is used. We have to show that the image $\mathfrak{g}_S$ of $\mathfrak{g}$ in $T_S\operatorname{Sec}(M)$ is non-degenerate for all $S \in \boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$. This is equivalent to

$$T_S\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0) \cap \operatorname{Mat}(2)\mathfrak{g}_S = \mathfrak{g}_S. \tag{7.3}$$

Indeed, $\mathfrak{g}_S$ is non-degenerate if and only if the quotient $T_S\operatorname{Sec}(M)/\operatorname{Mat}(2)\mathfrak{g}_S$ is trisymplectic (Claim 3.5). By (4.4), $T_S\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ is an orthogonal complement of $I\mathfrak{g}_S+J\mathfrak{g}_S+K\mathfrak{g}_S$ with respect to the holomorphic Riemannian form B associated with the trisymplectic structure, where I, J, K is some quaternionic basis in $\operatorname{Mat}(2)$. If $\mathfrak{g}_S$ is non-degenerate, the orthogonal complement of $\operatorname{Mat}(2)\mathfrak{g}_S$ is isomorphic to $T_S\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/\mathfrak{g}_S$, which gives (7.3). Conversely, if (7.3) holds, the orthogonal complement of $I\mathfrak{g}_S+J\mathfrak{g}_S+K\mathfrak{g}_S$ does not intersect $I\mathfrak{g}_S+J\mathfrak{g}_S+K\mathfrak{g}_S$, hence the restriction of B to $I\mathfrak{g}_S + J\mathfrak{g}_S + K\mathfrak{g}_S$ is non-degenerate. Therefore, non-degeneracy of $\mathfrak{g}_S$ is implied by Remark 3.7.

Now, let $S \in \operatorname{Sec}_0(M)$ be a twistor section which satisfies $\boldsymbol{\mu}_{\mathbb{R}}(S) = \boldsymbol{\mu}_{\mathbb{C}}(S) = 0$. By Proposition 6.1, for each $L \in \mathbb{C}P^1$, the corresponding point (L, S_L) of S satisfies $\mu_{hk}(S_L) = 0$, where $\mu_{hk}\ M \longrightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$ denotes the hyperkähler moment map. Let $g \in \operatorname{Mat}(2)\mathfrak{g}_S \cap T_S\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ be a vector obtained as a linear combination $\sum H_i g_i$, with $g_i \in \mathfrak{g}_S$ and $H_i \in \operatorname{Mat}(2, \mathbb{C})$. At each point $(L, S_L) \in S$, g is evaluated to a linear combination $\sum H_i^L g_i^L$ with quaternionic coefficients, tangent to $\mu_{hk}^{-1}(0)$. However, a quaternionic linear combination of this form can be tangent to $\mu_{hk}^{-1}(0)$ only if all H_i^L are real, because for each hyperkähler manifold Z one has a decomposition $T_x(\mu_{hk}^{-1}(0)) \oplus I\mathfrak{g} \oplus J\mathfrak{g} \oplus K\mathfrak{g} = TxZ$. We have proved that any $g \in \operatorname{Mat}(2)\mathfrak{g}_S \cap T_S\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)$ belongs to the image of $\mathfrak{g}$ at each point $(L, S_L) \in S$. This proves (7.3), hence, non-degeneracy of $\mathfrak{g}_S$.

Step 1: We prove that the image $\tau(\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G)$ belongs to $\operatorname{Sec}_0(M/\!\!/\!\!/G) \subset \operatorname{Sec}(M/\!\!/\!\!/G)$. Given $S \in \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$, consider its image $\tau(S)$ as a curve in

$\operatorname{Tw}(M/\!\!/\!\!/G)$, and let $N(\tau(S))$ be its normal bundle. Denote by $\tilde{S} \in \operatorname{Sec}_0(M)$ the twistor section which satisfies $\boldsymbol{\mu}_{\mathbb{R}}(\tilde{S}) = 0, \boldsymbol{\mu}_{\mathbb{C}}(\tilde{S}) = 0$ and gives S after taking a quotient. Then

$$N(\tau(S))\big|_L = \frac{T_{\mathsf{ev}_L(\tilde{S})}(M,L)}{\langle \mathfrak{g} + I\mathfrak{g} + J\mathfrak{g} + K\mathfrak{g}\rangle} \tag{7.4}$$

where $\mathsf{ev}_L : \operatorname{Sec}(M) \longrightarrow (M,L)$ is the standard evaluation map.

A bundle $B \cong \bigoplus_{2n} \mathcal{O}(1)$ can be constructed from a quaternionic vector space W as follows. For any $L \in \mathbb{CP}^1$, considered as a quaternion satisfying $L^2 = -1$, one takes the complex vector space (W,L) as a fiber of B at L. Denote this bundle as $B(W)$. Now, (7.4) gives

$$N(\tau(S)) = \frac{N(\tilde{S})}{B(\langle \mathfrak{g} + I\mathfrak{g} + J\mathfrak{g} + K\mathfrak{g}\rangle)}$$

giving a quotient of $\bigoplus_{2i} \mathcal{O}(1)$ by $\bigoplus_{2j} \mathcal{O}(1)$, which is also a direct sum of $\mathcal{O}(1)$. Therefore, $\tau(S)$ is regular.

Step 2: The tautological map $\tau : \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}(M/\!\!/\!\!/G)$ is a local diffeomorphism. This follows from the implicit function theorem. Indeed, let $S \in \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$ be a point associated with $\tilde{S} \in \operatorname{Sec}_0(M)$, satisfying $\boldsymbol{\mu}_{\mathbb{R}}(\tilde{S}) = 0, \boldsymbol{\mu}_{\mathbb{C}}(\tilde{S}) = 0$ as in Step 1. Then the differential of τ is a map

$$d\tau : \frac{\Gamma(N\tilde{S})}{\operatorname{Mat}(2,\mathbb{C}) \cdot \mathfrak{g}} \longrightarrow \Gamma(N\tau(S)), \tag{7.5}$$

where $\mathfrak{g} = \operatorname{Lie}(G) \subset T\operatorname{Tw}(M)$. Let $N_g\tilde{S}$ be a sub-bundle of $N\tilde{S}$ spanned by the image of $\langle \mathfrak{g} + I\mathfrak{g} + J\mathfrak{g} + K\mathfrak{g}\rangle$ By Step 1, $N_g S \cong \mathcal{O}(1)^k$, and, indeed, a subspace of $\Gamma(N\tilde{S})$ generated by $\operatorname{Mat}(2,\mathbb{C}) \cdot \mathfrak{g}$ coincides with $\Gamma(N_g\tilde{S})$. Similarly, $\Gamma(N\tau(S)) \cong \Gamma(N\tilde{S}/N_g\tilde{S})$. We have shown that the map (7.5) is equivalent to

$$\frac{\Gamma(N\tilde{S})}{\Gamma(N_g\tilde{S})} \longrightarrow \Gamma(N\tilde{S}/N_g\tilde{S}).$$

By step 1, the bundles $N\tilde{S}$ and $N_g\tilde{S}$ are sums of several copies of $\mathcal{O}(1)$, hence this map is an isomorphism.

Step 3: We prove that τ is compatible with the trisymplectic structure. The trisymplectic structure on $\operatorname{Sec}(M/\!\!/\!\!/G)$ is induced by a triple of holomorphic symplectic forms $\langle \mathsf{ev}_I^*(\Omega_I), \mathsf{ev}_J^*(\Omega_J), \mathsf{ev}_K^*(\Omega_K)\rangle$ (Claim 5.4). From the construction in Theorem 4.9 it is apparent that the same triple generates the trisymplectic structure on $\operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G$. Therefore, τ is compatible with the trisymplectic structure. We proved Proposition 7.1. ■

7.2 Trihyperkähler reduction and homogeneous bundles on $\mathbb{CP}^1$

Let M be a hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler isometries, and admitting a hyperkähler moment map. Consider

the set $Z \subset \operatorname{Tw}(M)$ consisting of all points $(m, L) \in \operatorname{Tw}(M)$ such that the corresponding holomorphic moment map vanishes on m: $\mu_L^{\mathbb{C}}(m) = 0$. By construction, Z is a complex subvariety of $\operatorname{Tw}(M)$. Let $G_{\mathbb{C}}$ be a complexificaion of G, acting on $\operatorname{Tw}(M)$ in a natural way, and $G_{\mathbb{C}} \cdot (m, L)$ its orbit. This orbit is called **stable**, if $G_{\mathbb{C}} \cdot m \subset (M, L)$ intersects the zero set of the real moment map,

$$G_{\mathbb{C}} \cdot m \cap \left(\mu_L^{\mathbb{R}}\right)^{-1}(0) \neq \emptyset.$$

As follows from the standard results about Kähler reduction, the union $Z_0 \subset Z$ of stable orbits is open in Z, and the quotient $Z_0/G_{\mathbb{C}}$ is isomorphic, as a complex manifold, to $\operatorname{Tw}(M /\!\!/\!\!/ G)$. Consider the corresponding quotient map,

$$P: \ Z_0 \longrightarrow Z_0/G_{\mathbb{C}} = \operatorname{Tw}(M /\!\!/\!\!/ G). \tag{7.6}$$

For any twistor section $S \in \operatorname{Sec}(M /\!\!/\!\!/ G)$, consider its preimage $P^{-1}(S)$. Clearly, $P^{-1}(S)$ is a holomorphic homogeneous vector bundle over $S \cong \mathbb{CP}^1$. We denote this bundle by P_S.

Proposition 7.2. *Let M be a flat hyperkähler manifold, and G a compact Lie group acting on M by hyperkähler isometries, and admitting a hyperkähler moment map. Consider the tautological map $\tau: \operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G \longrightarrow \operatorname{Sec}_0(M /\!\!/\!\!/ G)$ constructed in Proposition 7.1. Given a twistor section $S \in \operatorname{Sec}_0(M /\!\!/\!\!/ G)$, let P_S be a holomorphic homogeneous bundle constructed above. Then*

(i) *The point S lies in $\operatorname{Im} \tau$ if and only if the bundle P_S admits a holomorphic section (this is equivalent to P_S being trivial).*

(ii) *The map $\tau: \operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G \longrightarrow \operatorname{Sec}_0(M /\!\!/\!\!/ G)$ is an open embedding.*

Proof: A holomorphic section S_1 of P_S can be understood as a point in $\operatorname{Sec}(M)$. Since S_1 lies in the unio n of all stable orbits, denoted earlier as $Z_0 \subset Z \subset \operatorname{Tw}(M)$, the real moment map $\mu_L^{\mathbb{R}}$ is constant on S_1 (Proposition 6.1). By definition of Z_0, for each $(z, L) \in Z_0$, there exists $g \in G_{\mathbb{C}}$ such that $\mu_L^{\mathbb{R}}(gz) = 0$.

Therefore, $\boldsymbol{\mu}_{\mathbb{R}}(gS_1) = 0$ for appropriate $g \in G_{\mathbb{C}}$. This gives $\tau(S_2) = S$, where $S_2 \in \operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G$ is a point corresponding to gS_1. Conversely, consider a point $S_2 \in \operatorname{Sec}_0(M) /\!\!/\!\!/\!\!/ G$, such that $\tau(S_2) = S$, and let $S_1 \in S_0(M)$ be the corresponding twistor section. Then S_1 gives a section of P_S. We proved Proposition 7.2 (i).

To prove Proposition 7.2 (ii), it would suffice to show the following. Take $S \in \operatorname{Sec}_0(M /\!\!/\!\!/ G)$, and let $S_1, S_2 \in \operatorname{Sec}(M)$ be twistor sections which lie in Z_0 and satisfy $\boldsymbol{\mu}_{\mathbb{R}}(S_i) = 0$. Then there exists $g \in G$ such that $g(S_1) = S_2$. Indeed, $\tau^{-1}(S)$ is the set of all such S_i considered up to an action of G.

Let $P_S \xrightarrow{P} S$ be the homogeneous bundle constructed above, and $\mathcal{P}$ its fiber, which is a complex manifold with transitive action of $G_{\mathbb{C}}$. Using S_1, we trivialize $P_S = \mathcal{P} \times S$ in such a way that $S_1 = \{p\} \times S$ for some $p \in \mathcal{P}$. Then S_2 is a graph of a holomorphic map $\mathbb{CP}^1 \xrightarrow{\phi} \mathcal{P}$; to prove Proposition 7.2 (ii) it remains to show that ϕ is constant.

Since all points of $\left(\mu_L^{\mathbb{R}}\right)^{-1}(0)$ lie on the same orbit of G, the image $\phi(\mathbb{CP}^1)$ belongs to $G_p := G \cdot \{p\} \subset \mathcal{P}$. However, G_p is a totally real subvariety in $\mathcal{P} = G_{\mathbb{C}}/\operatorname{\mathsf{St}}(p)$. Indeed, G_p is fixed by a complex involution which exchanges the complex structure on $G_{\mathbb{C}}$ with its opposite. Therefore, all complex subvarieties of G_p are 0-dimensional, and $\phi: \mathbb{CP}^1 \longrightarrow G_p \subset \mathcal{P}$ is constant. We finished a proof of Proposition 7.2. ■

The proof of Theorem 5.11 follows. Indeed, by Proposition 7.1, the tautological map $\tau: \operatorname{Sec}_0(M)/\!\!/\!\!/\!\!/G \longrightarrow \operatorname{Sec}_0(M/\!\!/\!\!/G)$ is a local diffeomorphism compatible with the trisymplectic structures, and by Proposition 7.2 it is injective.

In particular, we have:

Corollary 7.3. *Let M be a flat hyperkähler manifold equipped with the action of a compact Lie group by hyperkähler isometries and admitting a hyperkähler moment map. Then the trihyperkähler reduction of* $\operatorname{Sec}_0(M)$ *is a smooth trisymplectic manifold of dimension* $2\dim M$.

■

8 Case study: moduli spaces of instantons

In this Section, we give an application of the previous geometric constructions to the study of the moduli space of framed instanton bundles on $\mathbb{CP}^3$. Our goal is to establish the smoothness of the moduli space of such objects, and show how that proves the smoothness of the moduli space of mathematical instanton bundles on $\mathbb{CP}^3$. That partially settles the long standing conjecture in algebraic geometry mentioned at the Introduction: moduli space of mathematical instanton bundles on $\mathbb{CP}^3$ of charge c is a smooth manifold of dimension $8c-3$, c. f. [CTT, Conjecture 1.2].

8.1 Moduli space of framed instantons on $\mathbb{R}^4$

We begin by recalling the celebrated ADHM construction of instantons, which gives a description of the moduli space of framed instantons on $\mathbb{R}^4$ in terms of a finite-dimensional hyperkähler quotient.

Let V and W be complex vector spaces of dimension c and r, respectively.

$$\mathbf{B} = \mathbf{B}(r,c) := \operatorname{End}(V) \oplus \operatorname{End}(V) \oplus \operatorname{Hom}(W,V) \oplus \operatorname{Hom}(V,W).$$

An point of $\mathbf{B}$ is a quadruple $X = (A,B,I,J)$ with $A,B \in \operatorname{End}(V)$, $I \in \operatorname{Hom}(W,V)$ and $J \in \operatorname{Hom}(V,W)$; it is said to be

(i) *stable* if there is no subspace $S \subsetneq V$ with $A(S), B(S), I(W) \subset S$;

(ii) *costable* if there is no subspace $0 \neq S \subset V$ with $A(S), B(S) \subset S \subset \ker J$;

(iii) *regular* if it is both stable and costable.

Let $\mathbf{B}^{\text{reg}}$ denote the (open) subset of regular data. The group $G = U(V)$ acts on $\mathbf{B}^{\text{reg}}$ in the following way:

$$g \cdot (A, B, I, J) := (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}). \tag{8.1}$$

It is not difficult to see that this action is free. The hyperkähler moment map $\mu : \mathbf{B}^{\text{reg}} \to \mathfrak{u}(V)^* \otimes \mathbb{R}^3$ can then be written in the following manner. Using the decomposition $\mathbb{R}^3 \simeq \mathbb{C} \oplus \mathbb{R}$ (as real vector spaces), we decompose $\mu = (\mu_{\mathbb{C}}, \mu_{\mathbb{R}})$ with $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ given by

$$\mu_{\mathbb{C}}(A, B, I, J) = [A, B] + IJ \quad \text{and} \tag{8.2}$$

$$\mu_{\mathbb{R}}(A, B, I, J) = [A, A^\dagger] + [B, B^\dagger] + II^\dagger - J^\dagger J. \tag{8.3}$$

The first component $\mu_{\mathbb{C}}$ is the holomorphic moment map $\mathbf{B} \to \mathfrak{gl}(V)^* \otimes_{\mathbb{R}} \mathbb{C}$ corresponding to the natural complex structure on $\mathbf{B}$.

The so-called *ADHM construction*, named after Atiyah, Drinfeld, Hitchin and Manin [ADHM], provides a bijection between the hyperkähler quotient $\mathcal{M}(r, c) := \mathbf{B}^{\text{reg}}(r, c) /\!\!/\!\!/ U(V)$ and framed instantons on the Euclidian 4-dimensional space $\mathbb{R}^4$; see [D] or [Nak, Theorem 3.48], and the references therein for details.

Let us now consider the trisymplectic reduction of $\text{Sec}_0(\mathbf{B}^{\text{reg}})$. As noted in the first few paragraphs of Section 6, $\text{Sec}_0(\mathbf{B}) = \text{Sec}(\mathbf{B}) \simeq \mathbf{B} \otimes \Gamma(\mathcal{O}_{\mathbb{CP}^1}(1))$, and $\text{Sec}_0(\mathbf{B}^{\text{reg}})$ is the (open) subset of $\text{Sec}_0(\mathbf{B})$ consisting of those sections σ such that $\sigma(p)$ is regular for every $p \in \mathbb{CP}^1$.

Definition 8.1. *A section $\sigma \in \mathbf{B} \otimes \Gamma(\mathcal{O}_{\mathbb{CP}^1}(1))$ is* globally regular *if $\sigma(p) \in \mathbf{B}$ is regular for every $p \in \mathbb{CP}^1$ (cf. [FJ, p. 2916-7], where such sections are called* $\mathbb{C}$-regular*).*

To be more precise, let $[z : w]$ be homogeneous coordinates on $\mathbb{CP}^1$; such choice leads to identifications

$$\Gamma(\mathcal{O}_{\mathbb{CP}^1}(1)) \simeq \mathbb{C}z \oplus \mathbb{C}w \simeq \mathbb{C}^2 \ \text{ and } \ \Gamma(\mathcal{O}_{\mathbb{CP}^1}(2)) \simeq \mathbb{C}z^2 \oplus \mathbb{C}w^2 \oplus \mathbb{C}zw \simeq \mathbb{C}^3. \tag{8.4}$$

It follows that $\text{Sec}_0(\mathbf{B}) \simeq \mathbf{B} \oplus \mathbf{B}$, so a point $\widetilde{X} \in \text{Sec}_0(\mathbf{B})$ can regarded as a pair (X_1, X_2) of ADHM data; $\widetilde{X}$ is globally regular (i.e. $\widetilde{X} \in \text{Sec}_0(\mathbf{B}^{\text{reg}})$) if any linear combination $zX_1 + wX_2$ is regular.

The action (8.1) of $GL(V)$ (hence also of $U(V)$) on $\mathbf{B}^{\text{reg}}$ extends to $\text{Sec}_0(\mathbf{B}^{\text{reg}})$ by acting trivially on the $\Gamma(\mathcal{O}_{\mathbb{CP}^1}(1))$ factor, i.e. $g \cdot (X_1, X_2) = (g \cdot X_1, g \cdot X_2)$.

Using the identification $\mathbb{C}^3 \simeq \Gamma(\mathcal{O}_{\mathbb{CP}^1}(2))$ above, it follows that the trisymplectic moment map $\boldsymbol{\mu}_{\mathbb{C}} : \text{Sec}_0(\mathbf{B}^{\text{reg}}) \to \mathfrak{u}(V)^* \otimes_{\mathbb{R}} \Gamma(\mathcal{O}_{\mathbb{CP}^1}(2))$ (constructed in Proposition 5.5) satisfies $\boldsymbol{\mu}_{\mathbb{C}}(\sigma)(p) = \mu_{\mathbb{C}}(\sigma(p))$ for $\sigma \in \text{Sec}_0(\mathbf{B}^{\text{reg}})$ and $p \in \mathbb{CP}^1$.

More precisely, let $X_1 = (A_1, B_1, I_1, J_1)$ and $X_2 = (A_2, B_2, I_2, J_2)$; consider the section $\sigma(z, w) = zX_1 + wX_2 \in \text{Sec}_0(\mathbf{B}^{\text{reg}})$. The identity $\boldsymbol{\mu}_{\mathbb{C}}(\sigma)(p) =$

$\mu_{\mathbb{C}}(\sigma(p))$ means that $\boldsymbol{\mu}_{\mathbb{C}}(\sigma)=0$ iff $\mu_{\mathbb{C}}(zX_1+wX_2)=0$ for every $[z:w]\in\mathbb{CP}^1$. Note that

$$\mu_{\mathbb{C}}(zX_1+wX_2)=0\Leftrightarrow\begin{cases}[A_1,B_1]+I_1J_1=0\\ [A_2,B_2]+I_2J_2=0\\ [A_1,B_2]+[A_2,B_1]+I_1J_2+I_2J_1=0\end{cases}\tag{8.5}$$

The three equations on the right hand side of equation (8.5) are known as the *1-dimensional ADHM equations*; they were first considered by Donaldson in [D] (cf. equations (a-c) in [D, p. 456]) and further studied in [FJ] (cf. equations (7-9) in [FJ, p. 2917]) and generalized in [J2, equation (3)].

One can show that globally regular solutions of the 1-dimensional ADHM equations are GIT-stable with respect to the $GL(V)$-action, see [HL, Section 3] and [HJV, Section 2.3]. Therefore, according to Remark 5.10, the trihyperkähler quotient $\mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}})/\!\!/\!\!/\!\!/U(V)$ is well defined and coincides with $\boldsymbol{\mu}_{\mathbb{C}}^{-1}(0)/GL(V)$.

8.2 Moduli space of framed instanton bundles on $\mathbb{CP}^3$

Recall that an *instanton bundle* on $\mathbb{CP}^3$ is a locally free coherent sheaf E on $\mathbb{CP}^3$ satisfying the following conditions

- $c_1(E)=0$;
- $H^0(E(-1))=H^1(E(-2))=H^2(E(-2))=H^3(E(-3))=0$.

The integer $c:=c_2(E)$ is called the *charge* of E. One can show that if E is an instanton bundle on $\mathbb{CP}^3$, then $c_3(E)=0$.

Moreover, a locally free coherent sheaf E on $\mathbb{CP}^3$ is said to be *of trivial splitting type* if there is a line $\ell\subset\mathbb{CP}^3$ such that the restriction $E|_\ell$ is the free sheaf, i.e. $E|_\ell\simeq\mathcal{O}_\ell^{\oplus\mathrm{rk}E}$. A *framing* on E at the line ℓ is the choice of an isomorphism $\phi:E|_\ell\to\mathcal{O}_\ell^{\oplus\mathrm{rk}E}$. A *framed bundle* (at ℓ) is a pair (E,ϕ) consisting of a locally free coherent sheaf E of trivial splitting type and a framing ϕ at ℓ. Two framed bundles (E,ϕ) and (E',ϕ') are isomorphic if there exists a bundle isomorphism $\Psi:E\to E'$ such that $\phi'=\phi\circ(\Psi|_\ell)$.

The following linear algebraic description of framed instanton bundles on $\mathbb{CP}^3$ was first established in [FJ, Main Theorem], and further generalized in [J2, Theorem 3.1].

Theorem 8.2. *The exists a 1-1 correspondence between equivalence classes of globally regular solutions of the* 1*-dimensional ADHM equations and isomorphism classes of instanton bundles on* $\mathbb{CP}^3$ *framed at a fixed line* ℓ*, where* $\dim W=\mathrm{rk}(E)\geq 2$ *and* $\dim V=c_2(E)\geq 1$.

Corollary 8.3. *The moduli space* $\mathcal{F}_\ell(r,c)$ *of rank* r*, charge* c *instanton bundles on* $\mathbb{CP}^3$ *framed at a fixed line* ℓ *is naturally identified with the trihyperkähler reduction* $\mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}}(r,c))/\!\!/\!\!/\!\!/U(V)$.

Proof. As we have shown above, the space $\mathcal{F}_\ell(r,c)$ is identified with the space of globally regular solutions of the 1-dimensional ADHM equation. In Subsection 8.1, we identified this space with $\mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}}(r,c))/\!\!/\!\!/\!\!/U(V)$. ■

We are finally in position to use Theorem 5.11 to obtain the second main result of this paper.

Theorem 8.4. *The moduli space $\mathcal{F}_\ell(r,c)$ of rank r, charge c instanton bundles on $\mathbb{CP}^3$ framed at a fixed line ℓ, is a smooth trisymplectic manifold of complex dimension $4rc$.*

Proof. The moduli space $\mathcal{M}(r,c) := \mathbf{B}^{\mathrm{reg}}(r,c)/\!\!/\!\!/U(V)$ of framed instantons of rank r and charge c is known to be a smooth, connected, hyperkähler manifold of complex dimension $2rc$; it follows that $\mathrm{Sec}_0(\mathcal{M}(r,c))$ is a smooth, trisymplectic manifold of complex dimension $4rc$. From [JV, Thm 3.8], we also know that the standard map

$$\mathcal{F}_\ell(r,c) \longrightarrow \mathrm{Sec}(\mathcal{M}(r,c)) \tag{8.6}$$

is an isomorphism (without the condition of regularity, which implies smoothness). From Theorem 8.3 it follows that $\mathcal{F}_\ell(r,c)$ is a trihyperkähler reduction of $\mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}}(r,c))$. ¿From its construction it is clear that the map (8.6) coincides with the map

$$\mathcal{F}_\ell(r,c) = \mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}}(r,c))/\!\!/\!\!/\!\!/U(V) \longrightarrow \mathrm{Sec}(\mathbf{B}^{\mathrm{reg}}(r,c)/\!\!/\!\!/U(V) = \mathrm{Sec}(\mathcal{M}(r,c)) \tag{8.7}$$

constructed in Theorem 5.11. ¿From this theorem it follows that (8.7) is in fact an open embedding to $\mathrm{Sec}_0(\mathcal{M}(r,c))$. Since (8.6) is an isomorphism,

$$\mathrm{Sec}_0(\mathcal{M}(r,c)) = \mathrm{Sec}(\mathcal{M}(r,c)).$$

This latter space is smooth, which proves smoothness of $\mathcal{F}_\ell(r,c)$. ■

Remark 8.5. Notice that Theorem 5.11 in itself only shows that the space $\mathcal{F}_\ell(r,c)$, which is a trihyperkähler reduction of $\mathrm{Sec}_0(\mathbf{B}^{\mathrm{reg}}(r,c))$, is openly embedded to $\mathrm{Sec}_0(\mathcal{M}(r,c))$. This already proves that $\mathcal{F}_\ell(r,c)$ is smooth, but to prove that this map is an isomorphism, we use [JV, Thm 3.8].

8.3 Moduli space of rank 2 instanton bundles on $\mathbb{CP}^3$

Let us now focus on the case of rank 2 instanton bundles, which is rather special. Recall that a mathematical instanton bundle on $\mathbb{CP}^3$ is a rank 2 stable bundle $E \to \mathbb{CP}^3$ with $c_1(E) = 0$ and $H^1(E(-2)) = 0$.

Proposition 8.6. *Rank 2 instanton bundles on $\mathbb{CP}^3$ are precisely mathematical instanton bundles.*

Proof. if E is a mathematical instanton bundle, then $H^0(E(-1)) = 0$ by stability. Since $\Lambda^2 E = \mathcal{O}_{\mathbb{CP}^3}$, there is a (unique up to a scalar) symplectic isomorphism between E and its dual E^*; one can then use Serre duality to show that $H^2(E(-2)) = H^3(E(-3)) = 0$, thus E is a rank 2 instanton bundle.

Conversely, every instanton bundle can be presented as the cohomology of a linear monad on $\mathbb{CP}^3$ [J1, Theorem 3]. It is then a classical fact that if E is a rank 2 bundle obtained as the cohomology of a linear monad on $\mathbb{CP}^3$ then E is stable. It is then clear that every rank 2 instanton bundle is a mathematical instanton bundle. ∎

Let $\mathcal{I}(c)$ denote the moduli space of mathematical instanton bundles and $\mathcal{I}_\ell(c)$ the open subset of $\mathcal{I}(c)$ consisting of instanton bundles restricting trivially to a fixed line $\ell \subset \mathbb{CP}^3$.

Let also $\mathcal{G}(c)$ denote the moduli space of S-equivalence classes of semistable torsion-free sheaves E of rank 2 on $\mathbb{P}^3$ with $c_1(E) = 0$, $c_2(E) = c$ and $c_3(E) = 0$; it is a projective variety. $\mathcal{I}(c)$ can be regarded as the open subset of $\mathcal{G}(c)$ consisting of those locally free sheaves satisfying $H^1(E(-1)) = 0$.

For any fixed line $\ell \subset \mathbb{CP}^3$, $\mathcal{I}(c)$ is contained in $\overline{\mathcal{I}_\ell(c)}$, where the closure is taken within $\mathcal{G}(c)$. Thus $\mathcal{I}(c)$ is irreducible if and only if there is a line ℓ such that $\mathcal{I}_\ell(c)$ is irreducible.

Using a theorem due to Grauert and Mullich we can conclude that every mathematical instanton bundle must restrict trivially at some line $\ell \subset \mathbb{CP}^3$ (see [JV, Lemma 3.12]). Therefore, $\mathcal{I}(c)$ is covered by open subsets of the form $\mathcal{I}_\ell(c)$, but it is not contained within any such sets, since for any nontrivial bundle over $\mathbb{CP}^3$ there must exist a line ℓ' such that the restricted sheaf $E|_{\ell'}$ is nontrivial. Thus $\mathcal{I}(c)$ and $\mathcal{I}_\ell(c)$ must have the same dimension, and one is nonsingular if and only if the other is as well.

We are now ready to prove the smoothness of the moduli space of mathematical instanton bundles on $\mathbb{CP}^3$.

Theorem 8.7. *The moduli space $\mathcal{I}(c)$ of mathematical instanton bundles on $\mathbb{CP}^3$ of charge c is a smooth complex manifold of dimension $8c - 3$.*

Proof. The forgetful map $\mathcal{F}_l(2,c) \to \mathcal{I}_\ell(c)$ that takes the pair (E,ϕ) simply to E has as fibers the the set of all possible framings at ℓ (up to equivalence). Since $E|_\ell \simeq W \otimes \mathcal{O}_\ell$ [FJ, Proposition 13], a choice of framing correspond to a choice of basis for the 2-dimensional space W, thus all fibers of the forgetfull map are isomorphic to $SL(W)$. Since $\mathcal{F}_l(2,c)$ is smooth of dimension $8c$, we conclude that $\mathcal{I}_\ell(c)$ is also smooth and its dimension is $8c-3$. The Theorem follows from our previous discussion. ∎

The irreducibility of $\mathcal{I}(c)$ for arbitrary c remains an open problem; it is only known to hold for c odd or $c = 2, 4$. Clearly, if $\mathcal{F}_l(2,c)$ is connected, then it must be irreducible, from which one concludes that $\mathcal{I}_\ell(c)$, and hence $\mathcal{I}(c)$, are irreducible. Since $\mathcal{F}_l(2,c)$ is a quotient of the set of globally regular solutions of the 1-dimensional ADHM equations, it is actually enough to prove that the latter is connected.

It is also worth mentioning a recent preprint of Markushevich and Tikhomirov [MT], in which the authours prove that $\mathcal{I}(c)$ is rational whenever it is irreducible. Thus, one also concludes immediately that $\mathcal{F}_l(2,c)$ is also rational whenever it is irreducible.

References

[AD] A. Andrada, I. G. Dotti, Double products and hypersymplectic structures on $\mathbb{R}^{4n}$. Comm. Math. Phys. **262** (2006), 1–16.

[ADHM] M. F. Atiyah, V. G. Drinfel'd, N. J. Hitchin, Yu. I. Manin, Construction of Instantons. Phys. Lett. A **65** (1978), 185–187.

[Ar] V. I. Arnold, The Lagrangian Grassmannian of a quaternion hypersymplectic space. Funct. Anal. Appl. **35** (2001), 61–63

[A] M. F. Atiyah, Complex analytic connections in fibre bundles. Trans. Amer. Math. Soc. **85** (1957), 181–207.

[B1] W. Barth, Some properties of stable rank-2 vector bundles on $\mathbf{P}_n$. Math. Ann. **226** (1977), 125–150.

[B2] W. Barth, Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$. Math. Ann. **258** (1981/82), 81–106.

[Bea] A. Beauville, Varietes Kähleriennes dont la première classe de Chern est nulle. J. Diff. Geom. **18** (1983), 755–782.

[Bes] A. Besse, *Einstein Manifolds*. Springer-Verlag, New York (1987).

[C] S. S. Chern, *Eine Invariantentheorie der Dreigewebe aus r-dimensionalen Mannigfaltigkeiten im R_{2r},* Abhandl. Math. Semin. Univ. Hamburg, 1936.

[CTT] I. Coandă, A. S. Tikhomirov, G. Trautmann, Irreducibility and smoothness of the moduli space of mathematical 5-instantons over $\mathbb{P}^3$. Internat. J. Math. **14** (2003), 1–45.

[DS] A. Dancer, A. Swann, Hypersymplectic manifolds. In: *Recent developments in pseudo-Riemannian geometry*, 97–111, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008.

[D] S. Donaldson, Instantons and Geometric Invariant Theory. Commun. Math. Phys. **93** (1984), 453–460.

[ES] G. Ellingsrud and S.A. Stromme, Stable rank 2 vector bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 3$. Math. Ann. **255** (1981), 123–135.

[F1] B. Feix, Hyperkähler metrics on cotangent bundles, J. Reine Angew. Math. **532** (2001), 33–46.

[FJ] I. B. Frenkel, M. Jardim, Complex ADHM equations, and sheaves on $\mathbb{P}^3$. J. Algebra **319** (2008), 2913–2937.

[GM] H. Grauert, G. Mülich, Vectorbundel von Rang 2 uber dem n-dimensionalen komplex projective Raum. Manuscripta Math. **16** (1975), 75–100.

[H] R. Hartshorne, Stable vector bundles of rank 2 on $\mathbb{P}^3$. Math. Ann. **238** (1978), 229–280.

[HL] M. Hauzer, A. Langer, Moduli spaces of framed perverse instantons on $\mathbb{P}^3$. Glasgow Math. J. **53** (2011), 51–96.

[HJV] A. A. Henni, M. Jardim, R. V. Martins, ADHM construction of perverse instanton sheaves. Preprint arXiv:1201:5657.

[HKLR] N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček, Hyperkähler metrics and supersymmetry. Comm. Math. Phys. **108**, (1987) 535–589.

[J1] M. Jardim, Instanton sheaves on complex projective spaces. Collec. Math. **57** (2006), 69–91.

[J2] M. Jardim, Atiyah–Drinfeld–Hitchin–Manin construction of framed instanton sheaves. C. R. Acad. Sci. Paris, Ser. I **346** (2008), 427–430.

[JV] M. Jardim, M. Verbitsky, Moduli spaces of framed instanton bundles on $\mathbb{CP}^3$ and twistor sections of moduli spaces of instantons on $\mathbb{R}^4$. Adv. Math. **227** (2011),1526–1538.

[K] D. Kaledin, Hyperkähler structures on total spaces of holomorphic cotangent bundles, in: D. Kaledin, M. Verbitsky (Eds.), Hyperkähler manifolds. International Press, Boston, 2001.

[KV] D. Kaledin, M. Verbitsky, Non-Hermitian Yang-Mills connections. Selecta Math. (N.S.) **4** (1998), 279–320.

[MT] D. Markushevich, A. S. Tikhomirov, Rationality of instanton moduli. Preprint arXiv:1012.4132.

[M] M. Maruyama, The Theorem of Grauert–Mülich–Spindler. Math. Ann. **255** (1981), 317–333.

[Nak] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces.* Providence: American Mathematical Society, 1999

[LeP] J. Le Potier, Sur l'espace de modules des fibrés de Yang et Mills. In: *Mathematics and physics*, 65–137. Progr. Math. **37**, Birkhuser Boston, 1983.

[Sal] S. Salamon, Quaternionic Kähler manifolds. Inv. Math. **67** (1982), 143–171.

[T] A. S. Tikhomirov. Moduli of mathematical instanton vector bundles with odd c_2 on projective space. Preprint arXiv:1101.3016.

[V1] M. Verbitsky, Hyperholomorphic bundles over a hyperkähler manifold. Journ. of Alg. Geom., **5** (1996), 633-669.

[V2] M. Verbitsky, Hypercomplex Varieties. Comm. Anal. Geom. **7** (1999), no. 2, 355–396.